# $p$-Adic $L$-functions

Maxwell Charles Siegel

Winter, 2018

siegelmaxwellc@ucla.edu; maxwelcs@usc.edu

## Section I: An Introduction to $p$-adic numbers

We owe the conception of the $p$-adic numbers to the mathematical work of Kurt Hensel in the 1890s. Although there are many ways of defining these numbers, the method I shall use—and, indeed, the method that I prefer over all others–is the very same approach that inspired Hensel to construct them in the first place: *power series*. In fact, many of the fundamental concepts associated with power series turn out to be the inspiration for much of the paradigm-shifting developments in number theory and algebraic geometry during the 20th century. Likely the most important of these is the simple observation that, in general, a power series for an analytic function $f : \mathbb{C} \to \mathbb{C}$ about a given point $z_0 \in \mathbb{C}$ gives us a formula for the function which is valid only in a *neighborhood* of $z_0$. Quite often, we need to compute power series expansion about different points (a process known as *analytic continuation*) in order to get formulae for the function on all the points in its domain. That is to say, the function itself is a *global* object, which we study *locally* (near the point $z_0$) by expanding it in a power series about a point.

**Definition**: Fix an integer $\nu \geq 2$ (not necessarily prime). Then, the set of **$\nu$-adic integers**, denoted $\mathbb{Z}_\nu$, is the set of all (formal) sums $x$ of the form:

$$x = c_0 + c_1\nu + c_2\nu^2 + \ldots$$

where the $c_n$s are elements of the set $\{0, 1, \ldots, \nu - 1\}$.

We can think of $\nu$-adic integers as "power series in $\nu$". Note that every non-negative integer is automatically a $\nu$-adic integer, seeing as every non-negative integer $x$ can be uniquely written as a *finite* sum $x = \sum_{n=0}^{N} c_n \nu^n$, where the $c_n$s are the **$\nu$-ary/adic digits** of $x$. The $\nu$-adic representation of $x \in \mathbb{Z}_\nu$ is the expression:

$$x = \mathbf{.}_\nu c_0 c_1 c_2 \ldots$$

where the subscript $\nu$ is there to remind us that we are in base $\nu$; this subscript can (and will) be dropped when there is no confusion as to the value of $\nu$.

$\mathbb{Z}_\nu$ becomes a commutative, unital ring when equipped with the usual addition and multiplication operations, albeit with the caveat that

$$c\nu^n = [c]_\nu \, \nu^n + \left(\frac{c - [c]_\nu}{\nu}\right) \nu^{n+1}$$

for all $n, c \in \mathbb{N}_0$. Ex: for $a \in \{0, \nu - 1\}$:

$$\centerdot_\nu 00 \left(\nu + a\right) = \left(\nu + a\right) \nu^2 = a\nu^2 + 1\nu^3 = \centerdot_\nu 00a1$$

where:

**Definition**: $[c]_\nu \in \{0, \ldots, \nu - 1\}$ denotes the residue class of $c$ mod $\nu$.

That is to say, like an odometer, *we carry over to the next $\nu$-adic digit's place whenever a digit reaches $\nu$.*

So, for example, in the 2-adics:

$$\begin{aligned} 3 &= 1 \times 2^0 + 1 \times 2^1 = \centerdot_2 11 \\ 5 &= 1 \times 2^0 + 0 \times 2^1 + 1 \times 2^2 = \centerdot_2 101 \end{aligned}$$

When we add these numbers, we "carry over the 2":

$$3 + 5 = \centerdot_2 11 + \centerdot_2 101 = \centerdot_2 211 = \centerdot_2 021 = \centerdot_2 002 = \centerdot_2 0001 = 1 \times 2^3 = 8$$

Multiplication is done similarly.

Using this arithmetic operation, we can write negative integers $\nu$-adically. For example, the $\nu$-adic number $y$ whose every $\nu$-adic digit is equal to $\nu - 1$:

$$y = \centerdot_\nu \left(\nu - 1\right) \left(\nu - 1\right) \left(\nu - 1\right) \left(\nu - 1\right) \ldots = \sum_{n=0}^{\infty} \left(\nu - 1\right) \nu^n$$

satisfies:

$$\begin{aligned} 1 + y &= \centerdot_\nu \left(\nu\right) \left(\nu - 1\right) \left(\nu - 1\right) \left(\nu - 1\right) \ldots \\ &= \centerdot_\nu 0 \left(\nu - 1 + 1\right) \left(\nu - 1\right) \left(\nu - 1\right) \ldots \\ &= \centerdot_\nu 0 \left(\nu\right) \left(\nu - 1\right) \left(\nu - 1\right) \ldots \\ &= \centerdot_\nu 00 \left(\nu - 1 + 1\right) \left(\nu - 1\right) \ldots \\ &= \centerdot_\nu 00 \left(\nu - 1 + 1\right) \left(\nu - 1\right) \ldots \\ &= \centerdot_\nu 000 \left(\nu - 1 + 1\right) \ldots \\ &\vdots \\ &= \centerdot_\nu 000000 \ldots \\ &= 0 \end{aligned}$$

and thus, $y = -1$ in $\mathbb{Z}_\nu$.

The beauty of the power series conception of these numbers is that it makes such formulae explicit:

$$\begin{aligned}
1+y &= 1+\sum_{n=0}^{\infty}(\nu-1)\,\nu^n \\
&= 1+\sum_{n=0}^{\infty}\nu^{n+1}-\sum_{n=0}^{\infty}\nu^n \\
&= \nu^0+\sum_{n=1}^{\infty}\nu^n-\sum_{n=0}^{\infty}\nu^n \\
&= \sum_{n=0}^{\infty}\nu^n-\sum_{n=0}^{\infty}\nu^n \\
&= 0
\end{aligned}$$

Consequently, just as when given a power series $\sum_{n=0}^{\infty} a_n z^n$, we can compute its multiplicative inverse $\sum_{n=0}^{\infty} b_n z^n$ recursively by way of the equations:

$$\begin{aligned}
1 &= \left(\sum_{m=0}^{\infty} a_m z^m\right)\left(\sum_{n=0}^{\infty} b_n z^n\right)=\sum_{k=0}^{\infty}\left(\sum_{n=0}^{k} a_n b_{k-n}\right) z^n \\
&\Updownarrow \\
1 &= a_0 b_0 \\
0 &= a_0 b_1 + a_1 b_0 \\
0 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \\
&\vdots
\end{aligned}$$

we can use the same formula to compute the multiplicative inverse of a given $\nu$-adic integer—*assuming* it exists. Working through with the above equations, it can be seen that the $\nu$-adic integer $x=\sum_{n=0}^{\infty} c_n \nu^n$ has a multiplicative inverse if and only if $c_0 \in \{0,\ldots,\nu-1\}$ is multiplicatively invertible mod $\nu$ (i.e., $c_0$ is co-prime to $\nu$). This is one of the reasons why we prefer to study $p$-adic integers, for prime $p$, simply because every non-zero residue class mod $p$ is then multiplicatively invertible modulo $p$.

**Definition**: Just as we can go from the ring of power series to the field of Laurent series, We can pass from the ring of $p$-adic integers $\mathbb{Z}_p$ to the field of **$p$-adic (rational) numbers** $\mathbb{Q}_p$ by considering "Laurent series" in $p$. Every $x \in \mathbb{Q}_p$ has a unique representation as:

$$x=\sum_{n=n_0}^{\infty} c_n p^n$$

for some $n_0 \in \mathbb{Z}$. We write the "fractional part" of $x$ (the terms with negative power of $p$) to the right of the $p$-adic point $\mathbf{.}_p$. Thus, the 3-adic number:

$$\frac{1}{3^2} + \frac{2}{3^1} + 0 \times 3^0 + 1 \times 3^1 + 1 \times 3^2 + 1 \times 3^3 + \dots$$

would be written as:

$$12 \, \mathbf{.}_3 \, 0\overline{1} = 12 \, \mathbf{.}_3 \, 0111\dots$$

where, as usual, the over-bar indicates that we keep writing 1 over and over again forever.

Now that we have a field—$\mathbb{Q}_p$—we can proceed to make field extensions by adjoining to $\mathbb{Q}_p$ the roots of irreducible polynomials with coefficients in $\mathbb{Q}_p$. Unfortunately, things are nowhere near as nice as with the ordinary rational numbers.

**Definition and Facts**: We write $\overline{\mathbb{Q}_p}$ (or $\overline{\mathbb{Q}}_p$) to denote the **algebraic closure** of $\mathbb{Q}_p$. Sadly, $\overline{\mathbb{Q}_p}$ is horrible: not only is it an *infinite dimensional* extension of $\mathbb{Q}_p$, but the metric space $\left(\overline{\mathbb{Q}_p}, \left|\cdot\right|_p\right)$ *fails* to be complete! As such, we write $\mathbb{C}_p$ (or $\Omega_p$) to denote the *metric completion* of $\left(\overline{\mathbb{Q}_p}, \left|\cdot\right|_p\right)$. $\mathbb{C}_p$ is called the field of **complex $p$-adic numbers**. This space is truly monstrous: it is *not* locally compact! Moreover, to make sense of it—namely, to prove that $\mathbb{C}_p$ is ring-isomorphic to $\mathbb{C}$—one must invoke the dreaded dark god of depravity and madness otherwise known as the **axiom of choice**.

**Definition**: We can equip $\mathbb{Q}_p$ with a metric space structure by defining the **$p$-adic valuation $\mathrm{val}_p$** (a.k.a. $\mathrm{ord}_p$, a.k.a. $\deg_p$) and **$p$-adic "norm" / absolute value** $\left|\cdot\right|_p$ by writing:

$$\mathrm{val}_p\left(\sum_{n=n_0}^{\infty} c_n p^n\right) \overset{\mathrm{def}}{=} n_0$$

$$\left|\sum_{n=n_0}^{\infty} c_n p^n\right|_p \overset{\mathrm{def}}{=} p^{-n_0}$$

that is, $\left|x\right|_p = p^{-\mathrm{val}_p(x)}$. The **$p$-adic metric** is then the distance formula defined by the map:

$$(x, y) \in \mathbb{Z}_p \times \mathbb{Z}_p \mapsto \left|x - y\right|_p$$

This metric is **non-archimedean**; that is, in addition to the triangle inequality, it satisfies the **strong triangle inequality**:

$$\left|x - y\right|_p \leq \max\left\{\left|x\right|_p, \left|y\right|_p\right\}$$

*Remark*: In the context of power series and Laurent series—say, a series $\sum_{n=n_0}^{\infty} c_n \left(z - z_0\right)^n$ about some $z_0 \in \mathbb{C}$—the $\nu$-adic valuation corresponds to the **zero degree** of the series at $z_0$. If $n_0$ is negative, then the function represented by that power series has a pole of order $-n_0$ at $z_0$; if $n_0$ is positive, then the function represented by that power series has a *zero* of order $n_0$ at $z_0$. That is, it tells us the number of times the function is divisible by the linear factor $z - z_0$. In terms of $\nu$-adic numbers, the $\nu$-adic valuation of a $\nu$-adic number tells us the amount of times that number is divisible by $\nu$. As defined, the $\nu$-adic metric assigns smaller sizes to $\nu$-adic numbers that are divisible by large powers of $\nu$. Thus, for all $n \in \mathbb{Z}$:

$$\left|\nu^n\right|_\nu = \nu^{-n}$$

This is especially important, seeing as the $\nu$-adic absolute value is a multiplicative group homomorphism from $\mathbb{Z}_\nu$ to $\mathbb{R}^+$:

$$\left|xy\right|_\nu = \left|x\right|_\nu \left|y\right|_\nu$$

as such, the level sets of $\mathbb{Z}_\nu$ are:

$$\left\{x \in \mathbb{Z}_\nu : |x| = \nu^{-n}\right\} = \nu^n \mathbb{Z}_\nu \overset{\text{def}}{=} \left\{\nu^n y : y \in \mathbb{Z}_\nu\right\}$$

for all $n \in \mathbb{N}_0$.

The bulk of the unusual properties of $p$-adic analysis stem from the non-archimedean nature of the $p$-adic topology.

Some examples: (Here, $K$ can be $\mathbb{Z}_p$ (or $\mathbb{Q}_p$) or any finite ring (or field) extension thereof.

- *Unlike in classical analysis*, a sequence $\left\{c_n\right\}_{n \in \mathbb{N}_0} \subseteq K$ is **Cauchy** if and only if $\lim_{n \to \infty} \left|c_{n+1} - c_n\right|_p = 0$.
- *Unlike in classical analysis*, a sequence $\left\{c_n\right\}_{n \in \mathbb{N}_0} \subseteq K$ is **summable** if and only if $\lim_{n \to \infty} \left|c_n\right|_p = 0$. Moreover, **all convergent series in *K* are absolutely convergent**—meaning that the sum is invariant under rearrangements of the terms of the summand.
- In $\mathbb{Z}_p$ or any finite extension thereof, all open sets are compact.
- All points inside a disk $D$ in $K$ are at the center of $D$.
- There exist differentiable functions $f : K \to \mathbb{C}_p$ which are injective but whose derivative $f'$ is identically 0.
- A continuous periodic function $f : K \to \mathbb{C}_p$ is **locally constant** (meaning that for every $x \in K$, there is an open neighborhood $U \subseteq K$ such that $f(a) = f(b)$ for all $a, b \in U$).

## Section II: An Introduction to Mellin Transforms, Dirichlet Series, and $L$-functions

**Definition**: Given a function $f : \mathbb{R}^+ \to \mathbb{C}$ the **Mellin Transform** of $f$, denoted $\mathscr{M}\left\{f\right\}$, is defined by the integral:

$$\mathscr{M}\left\{f\right\}(s) \overset{\text{def}}{=} \int_0^\infty x^{s-1} f(x)\, dx \tag{1}$$

for all complex numbers $s$ for which the integral is absolutely convergent. When the integral is convergent, there will exist real numbers $a$ and $b$ such that the integral converges (as written) for all $s$ with $a < \operatorname{Re}(s) < b$. The set of all such $s$ is called the **fundamental strip** of $\mathscr{M}\left\{f\right\}(s)$.

*Remark*: Performing the change of variables $e^{-t} = x$, we can write:

$$\mathscr{M}\left\{f\right\}(s) = \int_{-\infty}^{\infty} e^{-st} f\left(e^{-t}\right) du$$

which shows that the mellin transform of $f(t)$ is equal to the **bilateral Laplace transform** of $f\left(e^{-u}\right)$.

The real power of the Mellin transform lies not in the transform itself, but rather in its *inverse.*

**The Inverse Mellin Transform & The Mellin Inversion Theorem**: Formally, the **Inverse Mellin Transform** $\mathscr{M}^{-1}$ is defined by the following contour integral:

$$\mathscr{M}^{-1}\left\{F\right\}(x) \overset{\text{def}}{=} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s} ds$$

where $c$ is any real number in the fundamental strip of $F$.

*In particular*: Let $f : \mathbb{R}^+ \to \mathbb{C}$ be a function whose mellin transform, $F(s)$, has $a < \operatorname{Re}(s) < b$ as its fundamental strip, and which satisfies the properties:

i. $f(x)$ is piece-wise continuous on the positive real axis;

ii. $f(x)$ is of bounded variation on any interval $(p, q) \subseteq (0, \infty)$ on which it is continuous.

Next, for any $c \in (a, b)$ such that $F(s)$ has no singularities whose real part is *less* than $c$ (i.e., $c$ is to the right of all singularities of $F$ in its fundamental strip) and so that:

$$\int_{-\infty}^{\infty} \left|F(c + it)\right| dt < \infty$$

Then, for any $f$ and $c$ satisfying these conditions, we have that:

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s} ds = \begin{cases} f(x) & \text{if } f \text{ is continuous at } x \\ \frac{f(x^-)+f(x^+)}{2} & \text{if } f \text{ is discontinuous at } x \end{cases}, \ \forall x > 0 \tag{2}$$

where:

$$\begin{aligned} f\left(x^{+}\right) &\overset{\text{def}}{=} \lim_{t\downarrow x} f(t) \\ f\left(x^{-}\right) &\overset{\text{def}}{=} \lim_{t\uparrow x} f(t) \end{aligned}$$

The inverse mellin transform is almost *always* computed using the **Residue Theorem**. This is extraordinary, because it gives us an almost spectral-like characterization of the possible functions $f(x)$; namely, to recover the value of $f(x)$, all we need is information about the singularities of $\mathscr{M}\{f\}(s)$. In particular, *if it so happens that $\mathscr{M}\{f\}(s)$ has an analytic continuation to a region to the left of its fundamental strip, then, simply by knowing the nature of the singularities of $\mathscr{M}\{f\}(s)$ in that extra region, we can use the residue calculus to compute* ***explicit and/or asymptotic formulae*** *for $f(x)$*. In general, the further to the left we can continue $\mathscr{M}\{f\}(s)$, the more accurate the resultant formula for $f(x)$ will be. If we can continue $\mathscr{M}\{f\}(s)$ to a meromorphic function on $\mathbb{C}$, then, by taking into account all of the poles of $\mathscr{M}\{f\}(s)$ in the left half-plane adjacent to its fundamental strip, we can compute an explicit formula for $f(x)$.

As an example, considering the function:

$$\phi_r(x) \overset{\text{def}}{=} \sum_{n=0}^{\infty} x^{r^n}$$

for any real number $r > 1$, by applying the mellin transform to $\phi_r\left(e^{-x}\right)$, we obtain a meromorphic function:

$$\mathscr{M}\left\{\phi_r\left(e^{-x}\right)\right\}(s) = \int_0^{\infty} x^{s-1}\left(\sum_{n=0}^{\infty} e^{-r^n x}\right) dx = \sum_{n=0}^{\infty} \frac{\Gamma(s)}{r^{ns}} = \frac{\Gamma(s)}{1-r^s}$$

where $\Gamma(s)$ is the **Gamma Function**, defined by:

$$\Gamma(s) \overset{\text{def}}{=} \int_0^{\infty} x^{s-1} e^{-x} dx,\ \forall \operatorname{Re}(s) > 0 \tag{3}$$

Evaluating the inverse transform:

$$\mathscr{M}^{-1}\left\{\frac{\Gamma(s)}{1-r^s}\right\}(x) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \frac{\Gamma(s)}{1-r^s} x^{-s} ds$$

by integrating along the semicircular contour in the left half plane of infinite radius and with base along the line $\operatorname{Re}(s) = 1$, applying the Residue Theorem, and using the fact that the integrand vanishes along the arc of the semicircle, the following e*xact formula* can be obtained:

$$\phi_r\left(e^{-x}\right) = \frac{1}{2} - \frac{\gamma + \ln x}{\ln r} - \sum_{n=1}^{\infty} \frac{(-1)^n}{r^n - 1}\frac{x^n}{n!} + \frac{1}{\ln r}\sum_{k\in\mathbb{Z}^{\times}} \Gamma\left(\frac{2k\pi i}{\ln r}\right) e^{-2k\pi i \log_r x}\ \forall x > 0 \tag{4}$$

where $\gamma$ is the **Euler-Mascheroni constant**. (Here, $\phi_r\left(e^{-x}\right) \sim \frac{1}{2} - \frac{\gamma+\ln x}{\ln r} - \sum_{n=1}^{\infty} \frac{(-1)^n}{r^n-1} \frac{x^n}{n!}$ as $x \to \infty$; $\frac{1}{\ln r} \sum_{k\in\mathbb{Z}^\times} \Gamma\left(\frac{2k\pi i}{\ln r}\right) e^{-2k\pi i \log_r x}$ is a small, "fractal" function satisfying the functional equation $f(rx) = f(x)$ for all $x > 0$.) The reader will hopefully find it an informative exercise to perform the inverse transform for themselves.

Of particular import is the fact that, in performing the inverse transform, we obtain a formula that has a power series in $x$, whereas the original formula for $\phi_r\left(e^{-x}\right)$ is a power series in $e^{-x}$. This exponential change of variables is no accident. Historically, much (if not most) of analytic number theory owes its foundations to auspicious, seemingly-harmless identities in calculus. For example, for all $a \in \mathbb{R}^+$, we have the identity:

$$\frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} e^{-ax} dx = \frac{1}{a^s},\ \forall \mathrm{Re}(s) > 0 \tag{5}$$

(just substitute $y = ax$ to prove it). In terms of generating functions, this identity gives us a direct correspondence between **exponential power series**:

$$\sum_{n=1}^{\infty} c_n e^{-nx}$$

and **Dirichlet series**:

$$\sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

In general, one could say that the goal of analytic number theory is to figure out how to take a function defined by a series $\sum_{n=1}^{\infty} c_n e^{-nx}$ and then "untwist" it by writing it as a power series $\sum_{n=0}^{\infty} c_n' x^n$. In terms of generating functions of coefficients, power series in $x$ are easier to work with: the coefficients are simply the taylor coefficients of the function represented by the series. The same cannot be said of exponential power series. This is where Mellin transforms shine: by performing an inverse transform on $\mathscr{M}\left\{\sum_{n=1}^{\infty} c_n e^{-nx}\right\}(s)$, we will untwist the exponential power series $\sum_{n=1}^{\infty} c_n e^{-nx}$ into an ordinary power series, $\sum_{n=0}^{\infty} c_n' x^n$—or, at least, an asymptotic approximation of one.

Classically (as Dirichlet himself would have done), an $L$-function is a Dirichlet series of the form:

$$L(\chi, s) \overset{\text{def}}{=} \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},\ \forall \mathrm{Re}(s) > 1 \tag{6}$$

where $\chi$ is any Dirichlet character and $s$ is a complex variable.

Taking the right-hand side of this identity as an inspiration (seeing $n$ in the place of $a$), we then make the following computation using the above identity:

$$\begin{aligned}\Gamma(s)\,L(\chi,s) &= \sum_{n=1}^{\infty}\chi(n)\,\frac{\Gamma(s)}{n^s}\\ &= \sum_{n=1}^{\infty}\chi(n)\int_0^{\infty}x^{s-1}e^{-nx}dx\\ &= \int_0^{\infty}x^{s-1}\underbrace{\left(\sum_{n=1}^{\infty}\chi(n)\,e^{-nx}\right)}_{\text{call this } F_\chi(x)}dx\end{aligned}$$

Here, the interchange of $\sum$ and $\int$ is justified by the uniform convergence of $L(\chi,s)$ in $s$ for $\operatorname{Re}(s)>1$. Dividing by $\Gamma(s)$, we then obtain an integral transform representation of our $L$-function:

$$L(\chi,s)=\frac{1}{\Gamma(s)}\int_0^{\infty}x^{s-1}F_\chi(x)\,dx \tag{7}$$

Aside from being rather pretty to look at, this integral formula is extremely important, because it allows us to show that the $L$-function can be analytically continued with respect to $s$ into a meromorphic function on the complex plane. Not only that, but—when integrating by parts—it tells us an explicit formula for the values of $L(\chi,s)$ when $s$ is a negative integer.

**Proposition**: Let $f:\mathbb{R}^+\to\mathbb{C}$ be a smooth (infinitely differentiable) and bounded function satisfying $\lim_{x\to\infty}x^n f(x)=0$ for all $n\in\mathbb{N}_0$, and let:

$$M\{f\}(s)\overset{\text{def}}{=}\frac{1}{\Gamma(s)}\int_0^{\infty}x^{s-1}f(x)\,dx \tag{8}$$

Then, $M\{f\}(s)$ can be analytically continued to a meromorphic function on $\mathbb{C}$, and, in particular:

$$M\{f\}(-n)=(-1)^n f^{(n)}(0)\,,\ \forall n\in\mathbb{N}_0 \tag{9}$$

Proof: We simply integrate by parts. Utilizing the functional equation for the gamma function:

$$s\Gamma(s)=\Gamma(s+1)\,,\ \forall s\notin-\mathbb{N}_0 \tag{10}$$

the way forward is clear to us: we need to accrue an extra $s$, which will be produced by integrating $x^{s-1}$ with respect to $x$:

$$\begin{aligned}
M\{f\}(s) &= \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}f(x)\,dx \\
\left(\begin{array}{c} u=f(x)\,; dv=x^{s-1}dx \\ du=f'(x)\,; v=\frac{x^s}{s}\end{array}\right); &= \frac{1}{\Gamma(s)}\left(\left(\frac{x^s}{s}f(x)\right)|_{x=0}^{x=\infty}-\int_0^\infty \frac{x^s}{s}f'(x)\,dx\right) \\
&= \frac{1}{\Gamma(s)}\underbrace{\left(\frac{1}{s}\lim_{x\to\infty}x^s f(x)-\frac{0^s}{s}f(0)\right)}_{0,\ \forall \mathrm{Re}(s)>0}-\frac{1}{s\Gamma(s)}\int_0^\infty x^s f'(x)\,dx \\
&= -\frac{1}{s\Gamma(s)}\int_0^\infty x^s f'(x)\,dx \\
(s\Gamma(s)=\Gamma(s+1))\,; &= -\frac{1}{\Gamma(s+1)}\int_0^\infty x^{s+1-1}f'(x)\,dx \\
&= -M\{f'\}(s+1)
\end{aligned}$$

It can be shown that $f$ being infinitely differentiable guarantees that the given condition "$\lim_{x\to\infty} x^n f(x)=0$ for all $n\in\mathbb{N}_0$" forces "$\lim_{x\to\infty} x^n f^{(k)}(x)=0$ for all $n\in\mathbb{N}_0$" to be true for all $k\in\mathbb{N}_0$. As such, $M\{f'\}(s)$ converges for all $\mathrm{Re}(s)>0$. Thus, $-M\{f'\}(s+1)$ is therefore defined for all $\mathrm{Re}(s+1)>0$, i.e., for all $\mathrm{Re}(s)>-1$. This then gives an analytic continuation of $M\{f\}(s)$ to $\mathrm{Re}(s)>-1$.

To finish, we repeating the argument by induction. Integrating by parts $n$ times yields:

$$\begin{aligned}
M\{f\}(s) &= (-1)^n M\left\{f^{(n)}\right\}(s+n) \\
&= \frac{(-1)^n}{\Gamma(s+n)}\int_0^\infty x^{s+n-1}f^{(n)}(x)\,dx \\
(\text{pick } s=-n+1)\,; &\Downarrow \\
M\{f\}(-n+1)\,; &= \frac{(-1)^n}{\Gamma(1)}\int_0^\infty f^{(n)}(x)\,dx \\
(\Gamma(1)=(1-1)!=1)\,; &= (-1)^n\left(f^{(n-1)}(\infty)-f^{(n-1)}(0)\right) \\
&= (-1)^{n-1}f^{(n-1)}(0) \\
&\Updownarrow \\
M\{f\}(-(n-1)) &= (-1)^{n-1}f^{(n-1)}(0)
\end{aligned}$$

Finally, replacing $n$ with $n+1$ gives us the desired identity:

$$M\{f\}(-n)=(-1)^n f^{(n)}(0)\,,\ \forall n\in\mathbb{N}_0$$

Q.E.D.

Now, returning to the $L$-function:

$$L\left(\chi,s\right)=\frac{1}{\Gamma\left(s\right)}\int_{0}^{\infty}x^{s-1}F_{\chi}\left(x\right)dx$$

since $F_{\chi}\left(x\right)=\sum_{n=1}^{\infty}\chi\left(n\right)e^{-nx}$, and since $\chi\left(n\right)$ is a periodic function that takes finitely many distinct values, it can be shown that $F_{\chi}\left(x\right)$ will always be a rational function of $e^{x}$ with integer coefficients (i.e., $F_{\chi}\left(x\right)$ is the quotient of two polynomials in $e^{x}$ with integer coefficients), and thus, that $F_{\chi}\left(x\right)$ is exponentially decreasing—$x^{n}F_{\chi}\left(x\right)\rightarrow 0$ as $x\rightarrow\infty$ for all $n\in\mathbb{N}_{0}$. Consequently, our **Proposition** applies to the integral formula for $F_{\chi}\left(x\right)$, and we have that:

$$L\left(\chi,-k\right)=\left(-1\right)^{k}F_{\chi}^{\left(k\right)}\left(0\right)\tag{11}$$

This is the aforementioned "untwisting" in action:

$$F_{\chi}\left(x\right)=\sum_{n=1}^{\infty}\chi\left(n\right)e^{-nx}=\sum_{k=0}^{\infty}\frac{F_{\chi}^{\left(k\right)}\left(0\right)}{k!}x^{k}$$

Thus, in many respects, the real work-horse behind $L\left(\chi,s\right)$ is the function $F_{\chi}\left(x\right)$. This can be seen by the following *purely formal* computation of the "untwisted" formula for $F_{\chi}\left(x\right)$ as a power series in $x$:

$$\begin{aligned}
F_{\chi}\left(x\right) &= \sum_{n=1}^{\infty}\chi\left(n\right)e^{-nx}\\
\left(e^{z}=\sum_{k=0}^{\infty}\frac{z^{k}}{k!}\right);\quad &= \sum_{n=1}^{\infty}\chi\left(n\right)\sum_{k=0}^{\infty}\frac{\left(-nx\right)^{k}}{k!}\\
&= \sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{k!}\left(\sum_{n=1}^{\infty}\chi\left(n\right)n^{k}\right)x^{k}\\
\left(\sum_{n=1}^{\infty}\chi\left(n\right)n^{k}\equiv\sum_{n=1}^{\infty}\frac{\chi\left(n\right)}{n^{-k}}=L\left(\chi,-k\right)\right);\quad &= \sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{k!}L\left(\chi,-k\right)x^{k}
\end{aligned}$$

The snake-oil at work in this computation is our *re-interpretation* of the divergent series $\sum_{n=1}^{\infty}\chi\left(n\right)n^{k}$ as the value of our $L$-function at the negative integers. Of course, as our integration by parts argument showed, this snake-oil is in fact *perfectly justified*: the properties of the Mellin transform allow us to give a rigorous, analytic meaning to this formally divergent sum—namely, it is the analytic continuation of our $L$-function to the left half-plane.

## Section III: A Matter Of Measures - $p$-Adic Interpolation, Newton-Mahler Series, Measures on $\mathbb{Z}_p$ and the Amice-Mahler Transform

**$p$-Adic interpolation** is the art of extending classical functions to function defined over $\mathbb{Z}_p$. The central method of $p$-adic interpolation is to utilize density of $\mathbb{Z}$ in $\mathbb{Z}_p$ in the $p$-adic metric. Thus, given any function defined on $\mathbb{Z}$—say, $f : \mathbb{Z} \rightarrow \mathbb{C}$—we can attempt to define an extension $f^* : \mathbb{Z}_p \rightarrow \mathbb{C}$ (so that $f^*(n) = f(n)$ for all $n \in \mathbb{Z}$) at a given $x \in \mathbb{Z}_p$ by taking the limit (in $\mathbb{Z}_p$) of $f(x_n)$ as $n \rightarrow \infty$, where $\{x_n\}_{n \in \mathbb{N}_1} \subseteq \mathbb{Z}$ is any sequence of integers converging to $x$ in the $p$-adic topology; i.e., $\lim_{n \rightarrow \infty} |x - x_n|_p = 0$. To make this procedure work, however, there are some arithmetic caveats we must first take into account. Note that, by our construction, if $f^*$ exists, it will necessarily be continuous on $\mathbb{Z}_p$: for any $\{\mathfrak{y}_n\}_{n \in \mathbb{N}_0} \subseteq \mathbb{Z}_p$ converging to $\mathfrak{y} \in \mathbb{Z}_p$, the sequence $\{f^*(\mathfrak{y}_n)\}_{n \in \mathbb{N}_0}$ converges to $f(\mathfrak{y})$. Since the $p$-adic metric makes $\mathbb{Z}_p$ a *compact* metric space, we have that $f^*$, as a continuous function on $\mathbb{Z}_p$, is necessarily *uniformly continuous* on $\mathbb{Z}_p$: for any $\epsilon > 0$, there exists a $\delta > 0$ so that $|f^*(\mathfrak{y}) - f^*(\mathfrak{y}')| < \epsilon$ for all $\mathfrak{y}, \mathfrak{y}' \in \mathbb{Z}_p$ satisfying $|\mathfrak{y} - \mathfrak{y}'|_p < \delta$. Since the restriction $f^* |_{\mathbb{Z}}$ is equal to $f$, we have that for all $m, n \in \mathbb{Z}$, $|f(m) - f(n)|$ must be $< \epsilon$ whenever $|m - n|_p < \delta$. This tells us that not every function from $\mathbb{Z}$ to $\mathbb{C}$ can be interpolated $p$-adically; the function must already be $p$-adically continuous on $\mathbb{Z}$ for us to have any hope of interpolating it on $\mathbb{Z}_p$. Fortunately for us, $L$-functions turn out to have this property.

To begin with, we note that the Riemann Zeta Function $\zeta(s) = L(1, s)$ satisfies:

$$\zeta(-n) = \frac{(-1)^n}{n+1} \mathrm{b}_{n+1}, \ \forall n \in \mathbb{N}_0 \tag{12}$$

where $\mathrm{b}_{n+1}$ is the $(n+1)$th **Bernoulli number**. This follows by taking the formula:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = M\left\{\frac{1}{e^x - 1}\right\}(s) \tag{13}$$

and applying our integration by parts argument. Care is needed, though, in evaluating the boundary terms, seeing as $\frac{1}{e^x - 1}$ has a pole at $x = 0$. Modulo that detail, the connection between $\zeta(-n)$ and the Bernoulli numbers can be seen by the fact that:

$$\frac{x}{e^x - 1} = \sum_{n=1}^{\infty} \frac{\mathrm{b}_n}{n!} x^n = \mathrm{b}_0 + \mathrm{b}_1 x + \frac{\mathrm{b}_2}{2} x^2 + \frac{\mathrm{b}_3}{6} x^3 + \frac{\mathrm{b}_4}{24} x^4 + \cdots \tag{14}$$

The prophetic vision that eventually leads to the $p$-adic interpolation of $L$-functions is due to Kummer:

**Kummer's Congruences (1851)**: For any integers $m, n$, the congruence:

$$\frac{\mathrm{b}_m}{m} \overset{p}{\equiv} \frac{\mathrm{b}_n}{n} \tag{15}$$

occurs whenever $m \overset{p-1}{\equiv} n$ (where $\overset{c}{\equiv}$ means "congruence modulo $c$"). More generally, for any positive even integers $m, n$ which are not divisible by $p-1$ and any integer $a \geq 0$, the congruence:

$$\left(1-p^{m-1}\right) \frac{\mathrm{b}_m}{m} \overset{p^{a+1}}{\equiv} \left(1-p^{n-1}\right) \frac{\mathrm{b}_n}{n} \tag{16}$$

occurs whenever $m \overset{\varphi\left(p^{a+1}\right)}{\equiv} n$, where $\varphi$ is the **Euler Totient Function** ($\varphi(n)$ is the number of positive integers $\leq n$ which are co-prime to $n$), with $\varphi\left(p^{a+1}\right) = p^a(p-1)$.

Since $\zeta(-n) = (-1)^n \frac{\mathrm{b}_{n+1}}{n+1}$, we can write this as:

$$\left(1-p^{m-1}\right)(-1)^{m-1} \zeta(-(m-1)) \overset{p^{a+1}}{\equiv} \left(1-p^{n-1}\right)(-1)^{n-1} \zeta(-(n-1))$$

whenever $m \overset{\varphi\left(p^{a+1}\right)}{\equiv} n$. Since $\varphi\left(p^{a+1}\right)$ is even for all odd primes, we have that $m-n$, being divisible by $\varphi\left(p^{a+1}\right)$, is even for all odd primes, and thus, that:

$$\left(1-p^{m-1}\right) \zeta(1-m) \overset{p^{a+1}}{\equiv} \left(1-p^{n-1}\right) \zeta(1-n) \tag{17}$$

i.e., $p^{a+1}$ divides $\left|\left(1-p^{m-1}\right)\zeta(1-m) - \left(1-p^{n-1}\right)\zeta(1-n)\right|$, which is the same as writing:

$$\left|\left(1-p^{m-1}\right)\zeta(1-m) - \left(1-p^{n-1}\right)\zeta(1-n)\right|_p \leq \frac{1}{p^{a+1}}$$

Since $p^a \mid \varphi\left(p^{a+1}\right) \mid (m-n)$, we have that, for all odd primes and all $m, n \in \mathbb{Z}$:

$$|m-n|_p \leq \frac{1}{p^a} \Rightarrow \left|\left(1-p^{m-1}\right)\zeta(1-m) - \left(1-p^{n-1}\right)\zeta(1-n)\right|_p \leq \frac{1}{p^{a+1}} \tag{18}$$

which is to say, the function $\left(1-p^{n-1}\right)\zeta(1-n)$ is $p$-adically continuous in $n$.

This tells us that if we want to interpolate $\zeta$ (and hence, all $L$-functions) $p$-adically, we need to do something with the values of $\zeta$ at integers, both positive and negative. Although we can define a $p$-adic zeta function $\zeta_p$ using the aforementioned limiting method—with the validity of that approach being guaranteed by Kummer's congruences, as we just saw—it turns out to be much clearer if we instead utilize a measure theoretic approach. That there are multiple ways of realizing the $p$-adic analogue of an $L$-function is not at all surprising: $p$-adic analysis is a remarkably indecisive subject, especially for an area of analysis. Every time you find a way of defining something in it, there are probably other ways of doing so—all of them competing for the title of "best" or "least non-unique" approach.

Our adventure begins with the integral formula:

$$L(\chi, s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} F_\chi(x)\, dx$$

and the fact that:

$$L\left(\chi,-n\right)=\left(-1\right)^{n}F_{\chi}^{\left(n\right)}\left(0\right)$$

To get an inspiration, we need to have a minor stroke—courtesy of Ramanujan; certainly, G.H. Hardy must have nearly had one when he saw Ramanujan's "proof" of the following theorem.

**Ramanujan's Master Theorem**: Let $f$ be a function, and then consider its exponential generating function:

$$\varphi\left(x\right)=\sum_{n=0}^{\infty}\left(-1\right)^{n}\frac{f\left(n\right)}{n!}x^{n} \tag{19}$$

Then:

$$\int_{0}^{\infty}x^{s-1}\varphi\left(x\right)dx=\Gamma\left(s\right)f\left(-s\right) \tag{20}$$

Ramanujan's "Proof": We write out $\varphi$'s power series:

$$\int_{0}^{\infty}x^{s-1}\varphi\left(x\right)dx=\int_{0}^{\infty}x^{s-1}\sum_{n=0}^{\infty}\frac{\left(-x\right)^{n}}{n!}f\left(n\right)dx$$

Now, we have our stroke—*of genius!*

**Definition**: Consider the translation operator $\tau$ defined by:

$$\tau\left\{f\right\}\left(x\right)\overset{\text{def}}{=}f\left(x+1\right) \tag{21}$$

We write $\tau^{n}$ to denote the $n$th iterate of $\tau$; this satisfies:

$$\tau^{n}\left\{f\right\}\left(x\right)=f\left(x+n\right)$$

Returning to our "proof", we argue as follows:

$$\begin{aligned}\int_{0}^{\infty}x^{s-1}\sum_{n=0}^{\infty}\frac{\left(-x\right)^{n}}{n!}f\left(n\right)dx &= \int_{0}^{\infty}x^{s-1}\sum_{n=0}^{\infty}\frac{\left(-x\right)^{n}}{n!}\tau^{n}\left\{f\right\}\left(0\right)dx\\ &= \left(\int_{0}^{\infty}x^{s-1}\sum_{n=0}^{\infty}\frac{\left(-x\right)^{n}}{n!}\tau^{n}dx\right)\left\{f\right\}\left(0\right)\end{aligned}$$

That is to say, we are going to treat the integral as the integral of an operator $\int_{0}^{\infty}x^{s-1}\sum_{n=0}^{\infty}\frac{\left(-x\right)^{n}}{n!}\tau^{n}dx$ which acts upon the function $f$. Proceeding naïvely, we find that:

$$\begin{aligned}
\int_0^\infty x^{s-1}\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\tau^n dx &= \int_0^\infty x^{s-1}\sum_{n=0}^{\infty}\frac{(-\tau x)^n}{n!}dx \\
\left(\sum_{n=0}^{\infty}\frac{(-\tau x)^n}{n!}=e^{-\tau x}\right); &= \int_0^\infty x^{s-1}e^{-\tau x}dx \\
\left(y=\tau x; dx=\frac{dy}{\tau}\right); &= \int_0^\infty \left(\frac{y}{\tau}\right)^{s-1}e^{-y}\frac{dy}{\tau} \\
&= \tau^{-s}\underbrace{\int_0^\infty y^{s-1}e^{-y}dy}_{\Gamma(s)} \\
&= \Gamma(s)\,\tau^{-s}
\end{aligned}$$

Thus:

$$\begin{aligned}
\int_0^\infty x^{s-1}\varphi(x)\,dx &= \left(\int_0^\infty x^{s-1}\sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\tau^n dx\right)\{f\}(0) \\
&= \left(\Gamma(s)\,\tau^{-s}\right)\{f\}(0) \\
&= \Gamma(s)\left(\tau^{-s}\{f\}(0)\right) \\
\left(\tau^n\{f\}(0)=f(0+n)=f(n)\right); &= \Gamma(s)\,f(-s)
\end{aligned}$$

*Ta-daah!*

Note that if we divide by $\Gamma(s)$, we get:

$$f(-s)=\frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\varphi(x)\,dx$$

which looks very much like the integral formula for $L(\chi,s)$. But this is no accident! Recall how we wrote:

$$F_\chi(x)=\sum_{n=1}^{\infty}\chi(n)\,e^{-nx}=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}L(\chi,-k)\,x^k$$

Then, we see that $F_\chi(x)$ is the exponential generating function for the values of $L(\chi,s)$ at $s\in-\mathbb{N}_0$, and hence, our integration by parts argument from before actually proved a special case of Ramanujan's result!

The reason why this craziness matters has to do with the translation operator, $\tau$. We note that $\tau$ is a perfectly well-defined linear operator on spaces of continuous functions. Indeed, if we consider the $\mathbb{C}$-vector space $C^0(\mathbb{R},\mathbb{C})$ of continuous functions $f:\mathbb{R}\to\mathbb{C}$, we have that the map:

$$f\mapsto\tau\{f\}(0)$$

defines a linear functional on $C^0(\mathbb{R},\mathbb{C})$—that is, an element of the dual space $\left(C^0(\mathbb{R},\mathbb{C})\right)^*$. Not only that, but, note that we can even consider operators defined by polynomials in $\tau$:

$$\sum_{n=0}^{N} c_n \tau^n$$

which then induce the linear functionals:

$$f \mapsto \sum_{n=0}^{N} c_n \tau^n \{f\}(0) = \sum_{n=0}^{N} c_n f(n)$$

By doing as Ramanujan did and interpreting a function like $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} L(\chi,-k)\, x^k$ as the image of a function under a linear functional:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} L(\chi,-k)\, x^k &= \left(\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \tau^k x^k\right) \{L(\chi,\cdot)\}(-s) \\ &= \left(\sum_{k=0}^{\infty} \frac{(-\tau x)^k}{k!}\right) \{L(\chi,\cdot)\}(-s) \\ &= e^{-\tau x} \{L(\chi,\cdot)\}(-s) \end{aligned}$$

we will be able to realize *classical* $L$-functions as $p$-adic measures. A minor modification of these measures will then yield a bonafide $p$-adic $L$-function.

One of $p$-Adic analysis' more fascinating tendencies is how it strangely links the continuous and the discrete. Going all the way back to Isaac Newton, the **calculus of finite differences** deals with the discrete analogues of the operations studied in its more famous continuous cousin. Instead of an infinitesimal difference:

$$f'(z) \overset{\text{def}}{=} \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

it considers finite differences. Foremost among these is the forward difference operator, $\Delta$, defined by:

$$\Delta\{f\}(z) \overset{\text{def}}{=} f(z+1) - f(z) \tag{22}$$

This is sometimes denoted $f^{[1]}(z)$. More generally, iterating the forward difference $n$ times, we have the formula:

$$\Delta^n\{f\}(z) = f^{[n]}(z) \overset{\text{def}}{=} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(z+n-k) \tag{23}$$

where $f^{[0]}(z) = f(z)$).

Extending the binomial coefficient to the **binomial coefficient polynomials**:

$$\binom{z}{n} \overset{\text{def}}{=} \begin{cases} 1 & \text{if } n = 0 \\ \frac{z(z-1)\times\cdots\times(z-n+1)}{m!} & \text{if } n \geq 1 \end{cases} \tag{24}$$

a simple computation shows that, given a point $z_0 \in \mathbb{C}$, the formal series:

$$\mathcal{N}_{z_0}\left\{f\right\}(z) \overset{\text{def}}{=} \sum_{n=0}^{\infty} f^{[n]}(z_0)\binom{z-z_0}{n}$$

(the discrete analogue of the taylor series from continuous calculus) satisfies the formal identity:

$$\mathcal{N}_{z_0}\left\{f\right\}(z_0+n) = f(z_0+n) \ \forall n \in \mathbb{N}_0$$

If the series defining $\mathcal{N}_{z_0}\left\{f\right\}(z)$ converges for, say, all $\text{Re}(z) \geq 0$, then it defines a function, holomorphic on the right half-plane, with the property that $\mathcal{N}_{z_0}\left\{f\right\}(z_0+n) = f(z_0+n) \ \forall n \in \mathbb{N}_0$. Such a series is called the **Newton Series** of $f$; a **Theorem due to Carlson** gives conditions sufficient to guarantee that $\mathcal{N}_{z_0}\left\{f\right\}(z)$ and $f(z)$ represent the same function on the right half-plane. Not every holomorphic or meromorphic function has a convergent Newton series representation; however, many special functions (such as $L$-functions and zeta functions) *do* have such representations. In those cases where a function's Newton series converges to the function in question, the Newton series can be understood as being a holomorphic interpolation of the data set consisting of the input-output pairs $\left\{(z_0+n, f(z_0+n)) : n \in \mathbb{N}_0\right\}$.

Before continuing to the $p$-adic setting, however, we must return to our stroke of genius with $\tau$. Writing 1 to denote the identity operator on spaces of functions, observe that:

$$\Delta\left\{f\right\}(z) = f(z+1) - f(z) = \tau\left\{f\right\}(z) - f(z) = (\tau-1)\left\{f\right\}(z)$$

that is to say, we have the operator equation $\Delta = \tau - 1$. Equivalently: $\tau = 1 + \Delta$. Now, let's have another stroke—*of genius!*: pretending $\Delta$ is a number, suppose $|\Delta| < 1$. Then, *obviously*:

$$\ln(1+\Delta) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\Delta^n$$

is "convergent"—whatever that means. Since $1 + \Delta = \tau$, this means that we can (at least, formally) define the operator $\ln \tau$. To anyone that knows something about Lie Algebras, this should look very familiar. Letting $D$ denote the differentiation operator, we have that $\tau = e^D$ (i.e., the translation operator is the exponential of the differentiation operator, where $e^D$ is the operator $\sum_{n=0}^{\infty} \frac{D^n}{n!}$) and hence, that:

$$\tau = e^D = \Delta + 1 \tag{25}$$

The formula $\tau = e^D$ is nothing more than an operator-theoretic version of Taylor's Theorem—the fact that analytic functions are uniquely representable as power series). Indeed, for any entire function $f:\mathbb{C}\rightarrow\mathbb{C}$:

$$\begin{aligned}
e^{D}\left\{ f\right\} \left(z\right) & = \left(\sum_{n=0}^{\infty}\frac{D^{n}}{n!}\right)\left\{ f\right\} \left(z\right)\\
 & = \sum_{n=0}^{\infty}\frac{f^{\left(n\right)}\left(z\right)}{n!}\\
\left(\text{trick: }1=z+1-z=\left(z+1-z\right)^{n},\ \forall n\in\mathbb{N}_{0}\right); & = \underbrace{\sum_{n=0}^{\infty}\frac{f^{\left(n\right)}\left(z\right)}{n!}\left(\underbrace{z+1}_{\text{"}w\text{"}}-z\right)^{n}}_{\text{taylor series of }f\text{ @ }z\text{, eval. @ }w=z+1}\\
\left(\text{Fix }z\in\mathbb{C};\text{ then }f\left(w\right)=\sum_{n=0}^{\infty}\frac{f^{\left(n\right)}\left(z\right)}{n!}\left(w-z\right)^{n},\forall w\in\mathbb{C}\right); & = f\left(z+1\right)\\
 & = \tau\left\{ f\right\} \left(z\right)
\end{aligned}$$

This beautiful formula for the following reason—our second "stroke" of genius: in looking at an integral formula of the form $\frac{1}{\Gamma\left(s\right)}\int_{0}^{\infty}x^{s-1}F\left(x\right)dx$, we will treat appearances of $e^{x}$ in $F\left(x\right)$ as being instances of the translation operator $e^{D}$. The utility of this odd way of looking at things comes from the equation $e^{D}=\Delta+1$. The operator $e^{D}$ can only be applied to infinitely differentiable functions; even then, however, there is no guarantee that for a smooth function $\varphi$, $e^{D}\left\{ \varphi\right\} \left(x\right)$ will equal $\varphi\left(x+1\right)$, seeing as the validity of that identity (as shown in the above computation) requires $\varphi$ to be analytic on an open disk centered at $x$ of radius $>1$. However, to make sense of $\Delta\left\{ \varphi\right\} \left(x\right)$, all we need is for $\varphi$ to be continuous. Thus, by re-interpreting occurrences of $e^{D}$ in $F\left(x\right)$ as occurrences of the linear functional $1+\Delta$, we will be able to have a measure-theoretic interpretation of $F\left(x\right)$; that is, we will be able to view $F\left(x\right)$ as an operator on a space of continuous functions.

The miracle that makes this all of this hand-waving work lies in the peculiar properties of $p$-adic analysis. Here, interpolation reigns king; Newton series—now called **Mahler Series**—work for all continuous functions!

*Remark*: in order to avoid confusion between different types of variables, I will try to stick to using fraktur font (\mathfrak) for $p$-adic variables. The most common will be $\mathfrak{y}$ (\mathfrak{y}), because it looks much cooler than $\mathfrak{x}$ (\mathfrak{x}) or $\mathfrak{z}$ (\mathfrak{z}). I pronounce all fraktur letters by their english names (ex: $\mathfrak{y}$ is "y").)

**Mahler's Theorem**: Let $K$ be a finite extension of $\mathbb{Q}_{p}$ ($K$ can also be $\mathbb{Z}_{p}$, or $\mathbb{C}_{p}$, or any finite ring extension of $\mathbb{Z}_{p}$). Then, every continuous function $f:\mathbb{Z}_{p}\rightarrow K$ can be uniquely written as a **Mahler series**:

$$f\left(\mathfrak{y}\right)=\sum_{n=0}^{\infty}f^{\left[n\right]}\left(0\right)\binom{\mathfrak{y}}{n},\ \forall\mathfrak{y}\in\mathbb{Z}_{p}\tag{26}$$

which is convergent on all of $\mathbb{Z}_p$. Moreover, the **Mahler coefficients** of $f$, $\left\{f^{[n]}(0)\right\}_{n\in\mathbb{N}_0}\subseteq K$ satisfy:

$$\lim_{n\rightarrow\infty}\left|f^{[n]}(0)\right|_p=0$$

and:

$$\sup_{\mathfrak{y}\in\mathbb{Z}_p}\left|f(\mathfrak{y})\right|_p=\sup_{n\in\mathbb{N}_0}\left|f^{[n]}(0)\right|_p$$

*Remark*: This also works when $f$ is a continuous function from $\mathbb{Z}_p$ to a Banach space over $K$.

Example: Let's consider the function $f:\mathbb{Z}_p\rightarrow\mathbb{Z}_p$ defined by:

$$f(\mathfrak{y})\overset{\text{def}}{=}\mathfrak{y}^2$$

we observe that for any $\mathfrak{y},\mathfrak{y}'\in\mathbb{Z}_p$:

$$\begin{aligned} \left|f(\mathfrak{y}_N)-f(\mathfrak{y}'_N)\right|_p &= \left|\mathfrak{y}^2-\mathfrak{y}'^2\right|_p\\ &= \left|(\mathfrak{y}-\mathfrak{y}')(\mathfrak{y}+\mathfrak{y}')\right|_p\\ &= \left|\mathfrak{y}-\mathfrak{y}'\right|_p\left|\mathfrak{y}+\mathfrak{y}'\right|_p\\ \left(\mathfrak{y}-\mathfrak{y}'\in\mathbb{Z}_p\Rightarrow\left|\mathfrak{y}-\mathfrak{y}'\right|_p\leq 1\right);\quad &\leq \left|\mathfrak{y}-\mathfrak{y}'\right|_p \end{aligned}$$

Thus, $f(\mathfrak{y})=\mathfrak{y}^2$ is Lipschitz continuous on $\mathbb{Z}_p$, and is therefore continuous on $\mathbb{Z}_p$ As such, it has a Mahler series expansion. We compute the coefficients directly:

$$\begin{aligned} f^{[n]}(0) &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}f(n-k)\\ &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^2\\ (j=n-k);\quad &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{n-j}j^2\\ \left(\binom{n}{n-j}=\binom{n}{j}\right);\quad &= \sum_{j=0}^{n}(-1)^{n-j}\binom{n}{j}j^2 \end{aligned}$$

Noting that:

$$\begin{aligned}
(z-1)^n &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} z^j \\
\left(z\frac{d}{dz}\right); &\Downarrow \\
n(z-1)^{n-1} &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j z^{j-1} \\
\left(\times z, \frac{d}{dz}\right); &\Downarrow \\
\frac{d}{dz}\left\{zn(z-1)^{n-1}\right\} &= \frac{d}{dz}\left\{\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j z^j\right\} \\
&\Updownarrow \\
n(z-1)^{n-1} + n(n-1)z(z-1)^{n-2} &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^2 z^{j-1} \\
(\times z); &\Updownarrow \\
nz(z-1)^{n-1} + n(n-1)z^2(z-1)^{n-2} &= \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^2 z^j
\end{aligned}$$

We then evaluate at $z = 1$ to obtain:

$$n(1-1)^{n-1} + n(n-1)1^2(1-1)^{n-2} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^2$$

that is:

$$\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^2 = n\mathbf{1}_{\{1\}}(n) + n(n-1)\mathbf{1}_{\{2\}}(n) = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 0 & \text{else} \end{cases}$$

and so:

$$f^{[n]}(0) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} j^2 = \begin{cases} 1 & \text{if } n = 1 \\ 2 & \text{if } n = 2 \\ 0 & \text{else} \end{cases}$$

Thus, the Mahler Series of $f(\mathfrak{y}) = \mathfrak{y}^2$ is:

$$\mathfrak{y}^2 = \sum_{n=0}^{\infty} \binom{\mathfrak{y}}{n} f^{[n]}(0) = \binom{\mathfrak{y}}{1} + 2\binom{\mathfrak{y}}{2}$$

Indeed:

$$\begin{aligned}\binom{\mathfrak{y}}{1} &= \mathfrak{y}\\ \binom{\mathfrak{y}}{2} &= \frac{\mathfrak{y}\left(\mathfrak{y}-1\right)}{2}\end{aligned}$$

and so:

$$\binom{\mathfrak{y}}{1}+2\binom{\mathfrak{y}}{2}=\mathfrak{y}+\mathfrak{y}\left(\mathfrak{y}-1\right)=\mathfrak{y}+\mathfrak{y}^{2}-\mathfrak{y}=\mathfrak{y}^{2}$$

as desired.

Mahler's theorem is fundamental to everything that follows. For our purposes, it tells us that the binomial coefficient polynomials $\left\{\binom{\mathfrak{y}}{n}:n\in\mathbb{N}_{0}\right\}$ are a **basis** of $C^{0}\left(\mathbb{Z}_{p},K\right)$, the linear space of continuous functions from $\mathbb{Z}_{p}$ to $K$. Even better, using this basis, we then get a basis for $C^{0}\left(\mathbb{Z}_{p},K\right)$'s dual—$\left(C^{0}\left(\mathbb{Z}_{p},K\right)\right)^{*}$—*pro bono*. Since $\mathbb{Z}_{p}$ is a compact metric space, functional analysis tells us that $\left(C^{0}\left(\mathbb{Z}_{p},K\right)\right)^{*}$ is the space of all $J$-valued **Borel measures on** $\mathbb{Z}_{p}$ (where we consider a measure as a linear functional on $C^{0}\left(\mathbb{Z}_{p},K\right)$). Indeed, given a function $f\left(\mathfrak{y}\right)$, its coordinates with respect to the basis $\left\{\binom{\mathfrak{y}}{n}\right\}_{n\in\mathbb{N}_{0}}$ is exactly $\mathbb{N}$-tuple $\left\{f^{[n]}\left(0\right)\right\}_{n\in\mathbb{N}_{0}}$. As such, the dual basis is given by the linear maps $\left\{f\mapsto f^{[n]}\left(0\right)\right\}_{n\in\mathbb{N}_{0}}$. Writing $\Delta^{n}\left(f\right)\overset{\textrm{def}}{=}f^{[n]}\left(0\right)$, it then follows that the $\Delta^{n}$s form the dual basis for $\left(C^{0}\left(\mathbb{Z}_{p},K\right)\right)^{*}$:for all $f\left(\mathfrak{y}\right)=\sum_{n=0}^{\infty}c_{n}\binom{\mathfrak{y}}{n}\in C^{0}\left(\mathbb{Z}_{p},K\right)$. This proves the following corollary of Mahler's Theorem:

$$\Delta^{m}\left\{\sum_{n=0}^{\infty}c_{n}\binom{\mathfrak{y}}{n}\right\}=c_{m}$$

**Corollary ($\Delta$-Series formula for Measures on $C^{0}\left(\mathbb{Z}_{p},K\right)$):** Letting $K$ be a finite field extension of $\mathbb{Q}_{p}$, and letting $\Delta^{n}:C^{0}\left(\mathbb{Z}_{p},K\right)\rightarrow K$ denote the linear functionals

$$\Delta^{n}\left(f\right)\overset{\textrm{def}}{=}f^{[n]}\left(0\right)\overset{\textrm{def}}{=}\sum_{k=0}^{n}\left(-1\right)^{k}\binom{n}{k}f\left(n-k\right),\ \forall n\in\mathbb{N}_{0},\forall f\left(\mathfrak{y}\right)\in C^{0}\left(\mathbb{Z}_{p},K\right)$$

i.e.:

$$\Delta^{m}\left\{\sum_{n=0}^{\infty}c_{n}\binom{\mathfrak{y}}{n}\right\}=c_{m}\tag{27}$$

(that is, $\Delta^{m}$ gives us the $m$th coefficient of a function's Mahler series) we have that every measure/linear functional $d\mu\in\left(C^{0}\left(\mathbb{Z}_{p},K\right)\right)^{*}$ can be written as a $\Delta$-series of the form:

$$d\mu=\sum_{n=0}^{\infty}c_{n}\Delta^{n}\tag{28}$$

for some *unique* choice of constants $\{c_n\}_{n\in\mathbb{N}_0} \subseteq K$.

Knowing what $p$-adic measures look like then tells us how to evaluate them on functions in $C^0(\mathbb{Z}_p, K)$:

**Definition/Corollary.** (**Evaluation Formula for Measures on** $C^0(\mathbb{Z}_p, K)$): Given a function $f(\mathfrak{y}) \in C^0(\mathbb{Z}_p, K)$ and a measure $d\mu = \sum_{n=0}^{\infty} c_n \Delta^n \in \left(C^0(\mathbb{Z}_p, K)\right)^*$, we then have that the image of $f(\mathfrak{y})$ under $d\mu$—denoted $\int_{\mathbb{Z}_p} f(\mathfrak{y})\, d\mu(\mathfrak{y})$—is given by:

$$\int_{\mathbb{Z}_p} f(\mathfrak{y})\, d\mu(\mathfrak{y}) = \sum_{n=0}^{\infty} c_n \Delta^n(f) = \sum_{n=0}^{\infty} c_n f^{[n]}(0) \tag{29}$$

*Remark*: As usual, we can consider measures as additive set-functions on $\mathbb{Z}_p$ by writing:

$$|U|_{d\mu} \overset{\text{def}}{=} \int_{\mathbb{Z}_p} \mathbf{1}_U(\mathfrak{y})\, d\mu(\mathfrak{y}) = \int_U d\mu(\mathfrak{y})$$

where:

$$\mathbf{1}_U(\mathfrak{y}) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } \mathfrak{y} \in U \\ 0 & \text{if } \mathfrak{y} \notin U \end{cases}$$

is the indicator function of $U$.

In the language of linear functionals, all our previous hand-waving about $\Delta$, $\tau$, and $e^D$ can now be made fully rigorous. We see that, from the $p$-adic perspective, a measure on $C^0(\mathbb{Z}_p, K)$ is nothing but a power series in the operator $\Delta$ with coefficients in $K$. Remembering that $\Delta = e^D - 1$, we can—at least formally—then write such a measure as a power series in $e^D - 1$, which—if we then expand out and re-arrange the terms—then becomes a power series in $e^D$:

$$d\mu = \sum_{n=0}^{\infty} c_n \Delta^n = \sum_{n=0}^{\infty} c_n \left(e^D - 1\right)^n = \sum_{n=0}^{\infty} \tilde{c}_n e^{nD}$$

Crucially, note that if the $\tilde{c}_n$s form a periodic sequence (ex: $\tilde{c}_n = (-1)^n$), we can then write $\sum_{n=0}^{\infty} \tilde{c}_n e^{nD}$ as a rational function of $e^D$:

$$d\mu = \sum_{n=0}^{\infty} (-1)^n e^{nD} = \sum_{n=0}^{\infty} \left(-e^D\right)^n = \frac{1}{1+e^D}$$

This is important because of the fact that, for any complex number $s$ and any analytic function $f$:

$$e^{sD}\{f\}(z) = f(z+s)$$

Thus, whereas $e^D$ advances $f$ forward by 1, $e^{-D}$ makes $f$ step back by 1: $e^{-D}\{f\}(z) = f(z-1)$. This brings us back to our earlier observation that the measure we wanted to look for should have the property of being able to give us values of our $L$-functions at the negative integers!

$$L(\chi, s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( \sum_{n=1}^\infty \chi(n) e^{-nt} \right) dt$$

All of this swapping between power series in $x$ and power series in $e^{\pm t}$ is of no surprise to anyone familiar with mellin transform techniques in number theory and discrete mathematics. In the very broadest sense, the mellin transform (together with its inverse) is a device that allows us to re-write sufficiently well-behaved series of the form $\sum_{n=1}^\infty \tilde{c}_n e^{-nt}$ as series of the form $\sum_{n=0}^\infty c_n x^n$. This is of the *utmost* import for functions represented by a dirichlet series $\sum_{n=1}^\infty \frac{\tilde{c}_n}{n^s}$, because, then:

$$\sum_{n=1}^\infty \frac{\tilde{c}_n}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \left( \sum_{n=1}^\infty \tilde{c}_n e^{-nt} \right) dt$$

That is, the exponential series is the "easy" way to re-write the dirichlet series. The power-series representation, on the other hand, is "difficult" to obtain, and yet, it is far, far more valuable because it gives us direct information about the growth rate of the function under investigation. This *untangling* of exponential series into power series is directly paralleled by the relation between $e^D$ and $\Delta$ in the series representations of our measures. That is, the $\Delta$-series representation of $d\mu$ is analogous to the power series representation of the inverse mellin transform of a dirichlet series; on the other hand, the $e^D$ representation of $d\mu$ is analogous to the exponential series representation of the inverse mellin transform of a dirichlet series. The magic of the measure theoretic $p$-adic approach is that it allows us to make this analogy *rigorous*—in fact, this "analogy" is an *isomorphism of algebras.*

First, let us define a multiplication operation for measures:

**Definition**: For measures $d\mu, d\nu \in \left(C^0(\mathbb{Z}_p, K)\right)^*$, the **convolution of** $d\mu$ **and** $d\nu$ is the measure on $C^0(\mathbb{Z}_p, K)$ denoted by $d\mu * d\nu$ and defined by the formula:

$$\int_{\mathbb{Z}_p} f(\mathfrak{y}) (d\mu * d\nu)(\mathfrak{y}) \overset{\text{def}}{=} \int_{\mathbb{Z}_p} \left( \int_{\mathbb{Z}_p} f(\mathfrak{y} + \mathfrak{u}) \, d\nu(\mathfrak{u}) \right) d\mu(\mathfrak{y}) \tag{30}$$

Now that we have a multiplication operation, we can define the algebra of the hour:

**Definition**: The $K$**-Iwasawa Algebra** of $\mathbb{Z}_p$, denoted $\Lambda_K(\mathbb{Z}_p)$, is defined as the subspace of $\left(C^0(\mathbb{Z}_p, K)\right)^*$ consisting of all linear maps from $C^0(\mathbb{Z}_p, \mathcal{O}_K)$ (the space of continuous functions from $\mathbb{Z}_p$ to $\mathcal{O}_K$, the ring of integers of $K$) to $\mathcal{O}_K$:

$$\Lambda_K(\mathbb{Z}_p) \overset{\text{def}}{=} \mathcal{L}\left(C^0(\mathbb{Z}_p, \mathcal{O}_K), \mathcal{O}_K\right) \tag{31}$$

Equivalently, $\Lambda_K(\mathbb{Z}_p)$ is the set of measures for functions $f:\mathbb{Z}_p\rightarrow\mathcal{O}_K$ whose values on compact (equivalently, open) subsets of $\mathbb{Z}_p$ are integers in $K$; that is:

$$\Lambda_K(\mathbb{Z}_p)\overset{\text{def}}{=}\left\{d\mu\in\left(C^0(\mathbb{Z}_p,K)\right)^*:\int_{\mathbb{Z}_p}\mathbf{1}_U(\mathfrak{y})\,d\mu(\mathfrak{y})\in\mathcal{O}_K,\ \forall\text{ open }U\subseteq\mathbb{Z}_p\right\}$$

*Remark*: Since $\mathcal{O}_K\subseteq K$, we see that $\left(C^0(\mathbb{Z}_p,\mathcal{O}_K)\right)^*\subseteq\left(C^0(\mathbb{Z}_p,K)\right)^*$, and thus, just as every measure in $\left(C^0(\mathbb{Z}_p,K)\right)^*$ can be written as a $\Delta$-series, so too can every measure in the $K$-Iwasawa algebra. Not only that, but, these series will have coefficients in $\mathcal{O}_K$, and hence, the set of coefficients for any given measure will always be bounded in the $K$-adic valuation (because integers in $K$ have $K$-adic norm $\leq 1$). Thus, if we think of the $\Delta$ in the $\Delta$-series for elements of the Iwasawa algebra as a $p$-adic variable, these series will be *convergent* on $\mathbb{Z}_p$, and thus, will define *analytic functions* on $\mathbb{Z}_p$. This idea then suggests a way for us to make our analogy between $x$ and $\Delta$ explicit: come up with some way of converting the $\Delta$s in $d\mu$ into an $x$ in $\mathbb{Z}_p$. The following integral transform operator does this for us.

**Definition**: Given a measure $d\mu\in\Lambda_K(\mathbb{Z}_p)$, the **Amice-Mahler Transform (AMT)** of $d\mu$ is the $\mathcal{O}_K$-power series with bounded coefficients (in the variable $x$) denoted $\mathscr{A}\{d\mu\}(x)$, defined by:

$$\mathscr{A}\{d\mu\}(x)\overset{\text{def}}{=}\int_{\mathbb{Z}_p}(1+x)^{\mathfrak{y}}\,d\mu(\mathfrak{y})=\sum_{n=0}^{\infty}\left(\int\binom{\mathfrak{y}}{n}d\mu(\mathfrak{y})\right)x^n \tag{32}$$

To see what this transform does, let's consider an arbitrary measure $d\mu=\sum_{n=0}^{\infty}c_n\Delta^n\in\Lambda_K(\mathbb{Z}_p)$. I write $\Delta_{\mathfrak{y}}$ to remind us that $\Delta_{\mathfrak{y}}$ is the operator $\Delta:f\mapsto f(0)$ acting on $\mathfrak{y}$, and not $x$. Then:

$$\begin{aligned}
\mathscr{A}\{d\mu\}(x)\quad &= \int_{\mathbb{Z}_p}(1+x)^{\mathfrak{y}}\,d\mu(\mathfrak{y})\\
\left(\int_{\mathbb{Z}_p}f(\mathfrak{y})\,d\mu(\mathfrak{y})=\sum_{n=0}^{\infty}c_n\Delta_{\mathfrak{y}}^n(f)\right);\quad &= \sum_{n=0}^{\infty}c_n\Delta_{\mathfrak{y}}^n\left((1+x)^{\mathfrak{y}}\right)\\
\left((1+x)^{\mathfrak{y}}=\sum_{m=0}^{\infty}x^m\binom{\mathfrak{y}}{m}\right);\quad &= \sum_{n=0}^{\infty}c_n\Delta_{\mathfrak{y}}^n\underbrace{\left(\sum_{m=0}^{\infty}x^m\binom{\mathfrak{y}}{m}\right)}_{a_m=x^m}\\
\left(\Delta_{\mathfrak{y}}^n\left(\sum_{m=0}^{\infty}a_m\binom{\mathfrak{y}}{m}\right)=a_n\right);\quad &= \sum_{n=0}^{\infty}c_nx^n
\end{aligned}$$

Thus:

$$\mathscr{A}\left\{\sum_{n=0}^{\infty}c_n\Delta^n\right\}(x)=\sum_{n=0}^{\infty}c_nx^n \tag{33}$$

which is to say, the Amice transform takes the $\Delta$s in the series defining $d\mu$ and replaces them with the $p$-adic variable $x$.

Moreover, note that, by definition, $d\mu \in \Lambda_K(\mathbb{Z}_p)$ means that the coefficients $c_n$ of $d\mu$ belong to $\mathcal{O}_K$—the integers of $K$. Consequently, $\sup_{n\in\mathbb{N}_0} \left|c_n\right|_p$ is finite. This tells us that for every $d\mu$ in the $K$-Iwasawa algebra of $\mathbb{Z}_p$, $\mathscr{A}\{d\mu\}$ is a power series in the $p$-adic variable $x$ whose coefficients are bounded in $p$-adic norm.

The above formula for the output of the **AMT** immediately suggests how we can define the transform's inverse:

**Definition**: Given a function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ of $x \in \mathbb{Z}_p$ with bounded coefficients $\{c_n\}_{n\in\mathbb{N}_0} \subseteq \mathcal{O}_K$, the **Inverse Amice-Mahler Transform (IAMT)** of $f(x)$ (denoted $d\mu_f$ or $\mathscr{A}^{-1}\{f\}$) is defined by the measure:

$$d\mu_f = \mathscr{A}^{-1}\{f\} \overset{\text{def}}{=} \sum_{n=0}^{\infty} c_n \Delta^n \tag{34}$$

Thus, we see that the **AMT** and its inverse realizes our analogy: the transform replaces $\Delta$ in a measure with the $p$-adic variable $x$, while the inverse transform replaces $x$ with the forward difference functional $f \mapsto \Delta\{f\}(0)$.

**Theorem (Characterization of the $K$-Iwasawa algebra of $\mathbb{Z}_p$)**: The **AMT** $\mathscr{A} : \Lambda_K(\mathbb{Z}_p) \to \mathcal{O}_K^{\text{bdd}}[[x]]$ is an $\mathcal{O}_K$-algebra isomorphism between the $K$-Iwasawa algebra of $\mathbb{Z}_p$ and the ring of $\mathcal{O}_K$-power series with bounded coefficients.

Proof:

I. (Surjectivity) Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be an arbitrary function in $\mathcal{O}_K^{\text{bdd}}[[x]]$, convergent for all $x \in \mathbb{Z}_p$ for which $\left|x\right|_p < 1$. Note that the power series formula:

$$(1+x)^{\mathfrak{y}} = \sum_{m=0}^{\infty} \binom{\mathfrak{y}}{m} x^m$$

then holds for all $\mathfrak{y} \in \mathbb{Z}_p$ and all $\left|x\right|_p < 1$ (since, for every fixed $m$, $\binom{\mathfrak{y}}{m}$ is a continuous function of the $p$-adic variable $\mathfrak{y}$). For clarity, I write $\Delta_{\mathfrak{y}}$ to remind us that the $\Delta$ operator we are working with acts on function of $\mathfrak{y}$, rather than functions of $x$.

Then:

$$\begin{aligned}
\mathscr{A}\left\{d\mu_f\right\}(x) &= \underbrace{\int_{\mathbb{Z}_p}(1+x)^{\mathfrak{y}}\,d\mu_f(\mathfrak{y})}_{d\mu_f(\mathfrak{y})\text{ evaluated at }(1+x)^{\mathfrak{y}}}\\
\left(d\mu_f(\mathfrak{y})=\sum_{n=0}^{\infty}c_n\Delta_{\mathfrak{y}}^n\right); &= \sum_{n=0}^{\infty}c_n\Delta_{\mathfrak{y}}^n\left((1+x)^{\mathfrak{y}}\right)\\
\left((1+x)^{\mathfrak{y}}=\sum_{m=0}^{\infty}\binom{\mathfrak{y}}{m}x^m\right); &= \sum_{n=0}^{\infty}c_n\Delta_{\mathfrak{y}}^n\left(\sum_{m=0}^{\infty}\binom{\mathfrak{y}}{m}x^m\right)\\
\left(\Delta_{\mathfrak{y}}^n\text{ is linear}\right); &= \sum_{n=0}^{\infty}c_n\underbrace{\sum_{m=0}^{\infty}x^m\Delta_{\mathfrak{y}}^n\left(\binom{\mathfrak{y}}{m}\right)}_{=0\ \forall m\neq n}\\
\left(\Delta_{\mathfrak{y}}^n\left(\binom{\mathfrak{y}}{m}\right)=\begin{cases}1 & \text{if } n=m\\ 0 & \text{else}\end{cases}\right); &= \sum_{n=0}^{\infty}c_nx^n\\
&= f(x)
\end{aligned}$$

Thus, $\mathscr{A}$ is a surjective. ✓

II. (Injectivity) For any $d\mu=\sum_{n=0}^{\infty}a_n\Delta^n$ and $d\nu=\sum_{n=0}^{\infty}b_nx^n$ in $\Lambda_K(\mathbb{Z}_p)$, we have that:

$$\begin{aligned}
\mathscr{A}\{d\mu\} &= \mathscr{A}\{d\nu\}\\
&\Updownarrow\\
\sum_{n=0}^{\infty}a_nx^n &= \sum_{n=0}^{\infty}b_nx^n\\
&\Updownarrow\\
\sum_{n=0}^{\infty}(a_n-b_n)x^n &= 0\\
&\Updownarrow\\
a_n &= b_n,\ \forall n\in\mathbb{N}_0\\
\left(\left\{\Delta^n:n\in\mathbb{N}_0\right\}\text{ is a basis of }\Lambda_K(\mathbb{Z}_p)\right); &\Updownarrow\\
d\mu &= d\nu
\end{aligned}$$

and so, $\mathscr{A}$ is injective. ✓

III. (Algebra homomorphism) For any $d\mu=\sum_{n=0}^{\infty}a_n\Delta^n$ and $d\nu=\sum_{n=0}^{\infty}b_nx^n$ in $\Lambda_K(\mathbb{Z}_p)$, and any $\kappa\in\mathcal{O}_K$:

- $\mathcal{O}_K$-linearity:

$$\begin{aligned}
\mathscr{A}\left\{\kappa d\mu + d\nu\right\} &= \mathscr{A}\left\{\kappa\sum_{n=0}^{\infty} a_n\Delta^n + \sum_{n=0}^{\infty} b_n\Delta^n\right\} \\
&= \mathscr{A}\left\{\sum_{n=0}^{\infty}\left(\kappa a_n + b_n\right)\Delta^n\right\} \\
&= \sum_{n=0}^{\infty}\left(\kappa a_n + b_n\right)x^n \\
&= \kappa\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n \\
&= \kappa\mathscr{A}\left\{d\mu\right\} + \mathscr{A}\left\{d\nu\right\}
\end{aligned}$$

- $*$-homomorphicity:

$$\begin{aligned}
\mathscr{A}\left\{d\mu * d\nu\right\} &= \int_{\mathbb{Z}_p}(1+x)^{\mathfrak{y}}\left(d\mu * d\nu\right)(\mathfrak{y}) \\
&= \int_{\mathbb{Z}_p}\left(\int_{\mathbb{Z}_p}(1+x)^{\mathfrak{y}+\mathfrak{u}}\,d\nu(\mathfrak{u})\right)d\mu(\mathfrak{y}) \\
&= \int_{\mathbb{Z}_p}\left(\int_{\mathbb{Z}_p}(1+x)^{\mathfrak{y}}(1+x)^{\mathfrak{u}}\,d\nu(\mathfrak{u})\right)d\mu(\mathfrak{y}) \\
\left((1+x)^{\mathfrak{y}} \text{ is indep. of } \mathfrak{u}\right); \quad &= \int_{\mathbb{Z}_p}(1+x)^{\mathfrak{y}}\underbrace{\left(\int_{\mathbb{Z}_p}(1+x)^{\mathfrak{u}}\,d\nu(\mathfrak{u})\right)}_{\text{indep. of } \mathfrak{y}}d\mu(\mathfrak{y}) \\
&= \left(\int_{\mathbb{Z}_p}(1+x)^{\mathfrak{y}}\,d\mu(\mathfrak{y})\right)\left(\int_{\mathbb{Z}_p}(1+x)^{\mathfrak{u}}\,d\nu(\mathfrak{u})\right) \\
&= \mathscr{A}\left\{d\mu\right\}(x)\,\mathscr{A}\left\{d\nu\right\}(x)
\end{aligned}$$

Thus, $\mathscr{A}$ is a homomorphism of $\mathcal{O}_K$-algebras. ✓

Hence, $\mathscr{A}$ is an isomorphism of $\mathcal{O}_K$-algebras.

Q.E.D.

The power of the measure theoretic approach becomes evident once we realize that the underlying mechanics of the integration by parts formulae we saw in the classical setting remain valid in the $p$-adic setting. Now that we know how to integrate measures, we can proceed to learn how to integrate differential forms; that is, expressions of the form $g(\mathfrak{y})\,d\mu$, where $d\mu \in \Lambda_K(\mathbb{Z}_p)$ and $g(\mathfrak{y}) \in \mathcal{O}_K[[\mathfrak{y}]]$. Equivalently, we can equip the Iwasawa algebra with an **$\mathcal{O}_K[[\mathfrak{y}]]$-module structure**, as follows:

**Definition**: Given any $g(\mathfrak{y}) \in \mathcal{O}_K[[\mathfrak{y}]]$ and any $d\mu \in \Lambda_K(\mathbb{Z}_p)$, let $g(\mathfrak{y})\,d\mu$ denote the measure which acts upon functions $f(\mathfrak{y}) \in C^0(\mathbb{Z}_p, K)$ by the rule:

$$\int_{\mathbb{Z}_p} f(\mathfrak{y})\,(g(\mathfrak{y})\,d\mu)(\mathfrak{y}) \overset{\text{def}}{=} \int_{\mathbb{Z}_p} (f(\mathfrak{y})\,g(\mathfrak{y}))\,d\mu(\mathfrak{y}) \tag{35}$$

That is, the $\mathcal{O}_K[[\mathfrak{y}]]$-coefficient of $d\mu$ gets handed over to $f$.

Since the monomials $\{\mathfrak{y}^n\}_{n\in\mathbb{N}_0}$ generate $\mathcal{O}_K[[\mathfrak{y}]]$ as an $\mathcal{O}_K$-module (i.e., as an $\mathcal{O}_K$-"linear space"), by linearity, to understand the impact of multiplying a measure by a function, we need only understand what happens when we multiply a measure by $\mathfrak{y}^k$.

**Proposition**:

$$\mathscr{A}\{\mathfrak{y}\,d\mu\}(x) = (1+x)\frac{d}{dx}\{\mathscr{A}\{d\mu\}\}(x) \tag{36}$$

Proof: Observing the identities:

$$x\binom{x}{j} = (x-j)\binom{x}{j} + j\binom{x}{j} = (j+1)\binom{x}{j+1} + j\binom{x}{j}$$

we have that:

$$\begin{aligned}
\mathscr{A}\{\mathfrak{y}\,d\mu\}(x) &= \int_{\mathbb{Z}_p} \mathfrak{y}\,(1+x)^{\mathfrak{y}}\,d\mu(\mathfrak{y}) \\
&= \sum_{j=0}^{\infty}\left(\int_{\mathbb{Z}_p} \mathfrak{y}\binom{\mathfrak{y}}{j} d\mu(\mathfrak{y})\right) x^j \\
&= \sum_{j=0}^{\infty}\left(\int_{\mathbb{Z}_p} \left((j+1)\binom{\mathfrak{y}}{j+1} + j\binom{\mathfrak{y}}{j}\right) d\mu(\mathfrak{y})\right) x^j \\
&= \sum_{j=0}^{\infty}\left(\int_{\mathbb{Z}_p} (j+1)\binom{\mathfrak{y}}{j+1} d\mu(\mathfrak{y}) + \int_{\mathbb{Z}_p} j\binom{\mathfrak{y}}{j} d\mu(\mathfrak{y})\right) x^j \\
&= \sum_{j=0}^{\infty}(j+1)\left(\int_{\mathbb{Z}_p} \binom{x}{j+1} d\mu(\mathfrak{y})\right) x^j + \sum_{j=0}^{\infty} j\left(\int_{\mathbb{Z}_p} \binom{\mathfrak{y}}{j} d\mu(\mathfrak{y})\right) x^j \\
&= \sum_{j=1}^{\infty} j\left(\int_{\mathbb{Z}_p} \binom{\mathfrak{y}}{j} d\mu(\mathfrak{y})\right) x^{j-1} + x\sum_{j=1}^{\infty} j\left(\int_{\mathbb{Z}_p} \binom{\mathfrak{y}}{j} d\mu(\mathfrak{y})\right) x^{j-1} \\
&= \frac{d}{dx}\left\{\sum_{j=0}^{\infty}\left(\int_{\mathbb{Z}_p} \binom{\mathfrak{y}}{j} d\mu(\mathfrak{y})\right) x^j\right\} + x\frac{d}{dx}\left\{\sum_{j=0}^{\infty}\left(\int_{\mathbb{Z}_p} \binom{\mathfrak{y}}{j} d\mu(\mathfrak{y})\right) x^j\right\} \\
&= (1+x)\frac{d}{dx}\left\{\sum_{j=0}^{\infty}\left(\int_{\mathbb{Z}_p} \binom{\mathfrak{y}}{j} d\mu(\mathfrak{y})\right) x^j\right\} \\
&= (1+x)\frac{d}{dx}\{\mathscr{A}\{d\mu\}\}(x)
\end{aligned}$$

*Remark*: More generally, by induction, we have that:

$$\mathscr{A}\left\{\mathfrak{y}^{k}d\mu\right\}(x)=\left((1+x)\frac{d}{dx}\right)^{k}\{\mathscr{A}\{d\mu\}\}(x)\tag{37}$$

That is to say, the **AMT** converts multiplication by $\mathfrak{y}$ into an application of the differential operator $(1+x)\frac{d}{dx}$.

**Corollary**:

$$\int_{\mathbb{Z}_{p}}\mathfrak{y}^{k}d\mu_{f}(\mathfrak{y})=f^{(k)}(0)\tag{38}$$

Proof: We write out the identity:

$$\mathscr{A}\left\{\mathfrak{y}^{k}d\mu\right\}(x)=\left((1+x)\frac{d}{dx}\right)^{k}\{\mathscr{A}\{d\mu\}\}(x)$$

in integral form, obtaining:

$$\int_{\mathbb{Z}_{p}}\mathfrak{y}^{k}(1+x)^{\mathfrak{y}}d\mu(\mathfrak{y})=\left((1+x)\frac{d}{dx}\right)^{k}\{\mathscr{A}\{d\mu\}\}(x)$$

Letting $d\mu=d\mu_{f}$, we have that $\mathscr{A}\{d\mu_{f}\}(x)=f(x)$, and thus:

$$\int_{\mathbb{Z}_{p}}\mathfrak{y}^{k}(1+x)^{\mathfrak{y}}d\mu_{f}(\mathfrak{y})=\left((1+x)\frac{d}{dx}\right)^{k}\{f\}(x)$$

Finally, since $x$ is a $p$-adic variable, upon choosing $x=0$, we obtain:

$$\begin{array}{rcl}\displaystyle\int_{\mathbb{Z}_{p}}\mathfrak{y}^{k}1^{\mathfrak{y}}d\mu_{f}(\mathfrak{y}) & = & \displaystyle\left((1+0)\frac{d}{dx}\right)^{k}\{f\}(0)\\ & \Updownarrow & \\ \displaystyle\int_{\mathbb{Z}_{p}}\mathfrak{y}^{k}d\mu_{f}(\mathfrak{y}) & = & \displaystyle\frac{d^{k}}{dx^{k}}\{f\}(0)\\ & = & f^{(k)}(0)\end{array}$$

*Remark*: thus, in general, given a measure $d\mu$ and a power-series $f(\mathfrak{y})=\sum_{k=0}^{\infty}c_{k}\mathfrak{y}^{k}$, we have that:

$$\int_{\mathbb{Z}_{p}}f(\mathfrak{y})d\mu(\mathfrak{y})=\sum_{k=0}^{\infty}c_{k}\int_{\mathbb{Z}_{p}}\mathfrak{y}^{k}d\mu(\mathfrak{y})=\sum_{k=0}^{\infty}c_{k}\mathscr{A}\{d\mu\}^{(k)}(0)=\left(\sum_{k=0}^{\infty}c_{k}D^{k}\right)\{\mathscr{A}\{d\mu\}\}(0)$$

where $\mathscr{A}\{d\mu\}^{(k)}$ is the $k$th derivative of the **AMT** of $d\mu$.

More simply:

$$\int_{\mathbb{Z}_p} \mathfrak{y}^k d\mu(\mathfrak{y}) = \mathscr{A}\{d\mu\}^{(k)}(0) \tag{39}$$

*Remark*: The identities:

$$\underbrace{\mathscr{A}\left\{\mathfrak{y}^k d\mu\right\}(0)}_{\int_{\mathbb{Z}_p} \mathfrak{y}^k (1+0)^{\mathfrak{y}} d\mu(\mathfrak{y})} = \mathscr{A}\{d\mu\}^{(k)}(0)$$

and

$$\mathscr{A}\left\{\mathfrak{y}^k d\mu\right\}(x) = \left((1+x)\frac{d}{dx}\right)^k \{\mathscr{A}\{d\mu\}\}(x)$$

are the *raison d'être* of everything we have done this far. Indeed, this formula is directly connected to our analogy of $x \sim \Delta$ and $e^t \sim e^D$. To see this, we begin by relating $t$ and $x$ by way of the equation $e^t = x + 1$. This change of variables is the formal/concrete analogue of the operator identity $e^D = \Delta + 1$. The relevance of the **AMT** identity is that it describes the formula relating the derivative operators $\frac{d}{dt}$ and $\frac{d}{dx}$. Indeed, treating $x$ as a function of $t$, taking the total derivative of $e^t = x(t) + 1$ gives:

$$\begin{aligned}
\mathrm{d}\left\{e^t\right\} &= \mathrm{d}\{x+1\} \\
\frac{\partial}{\partial t}\left\{e^t\right\} dt &= \frac{\partial}{\partial t}\{x+1\} dx \\
e^t dt &= dx \\
(x+1) dt &= dx \\
(x+1)\frac{1}{dx} &= \frac{1}{dt} \\
(x+1)\frac{d}{dx} &= \frac{d}{dt}
\end{aligned}$$

Thus, the identity:

$$\mathscr{A}\left\{\mathfrak{y}^k d\mu\right\}(x) = \left((1+x)\frac{d}{dx}\right)^k \{\mathscr{A}\{d\mu\}\}(x)$$

can be written in terms of $t$ as:

$$\mathscr{A}\left\{\mathfrak{y}^k d\mu\right\}\left(e^t - 1\right) = \frac{d}{dt}\left\{\mathscr{A}\{d\mu\}\left(e^t - 1\right)\right\}$$

Thus, for a function $f(x) \in \mathcal{O}_K[[x]]$, upon considering the associated measure $d\mu_f = f(\Delta)$, we have that:

$$\begin{aligned}
\int_{\mathbb{Z}_p} \mathfrak{y}^k d\mu_f(\mathfrak{y}) &= \left((1+x)\frac{d}{dx}\right)^k \{f\}(x)\,|_{x=0} \\
\left(\text{let } e^t = x+1\right); &= \left(\frac{d}{dt}\right)^k \{f\}\left(e^t-1\right)|_{t=0} \\
\left(\text{let } F(t) \overset{\text{def}}{=} f\left(e^t-1\right)\right); &= \left(\frac{d}{dt}\right)^k \{F\}(t)\,|_{t=0} \\
&= F^{(k)}(0)
\end{aligned}$$

This reveals the truth behind our analogy. Noting that $F(t) = f(e^t-1)$ and $e^t = x+1$ imply $f(x) = F(\ln(1+x))$, we see that:

$$F^{(k)}(0) = \int_{\mathbb{Z}_p} \mathfrak{y}^k d\mu_f(\mathfrak{y}) = \int_{\mathbb{Z}_p} \mathfrak{y}^k f(\Delta_{\mathfrak{y}}) \approx \int_{\mathbb{Z}_p} \mathfrak{y}^k F(\ln(1+\Delta_{\mathfrak{y}}))$$

Since

$$\ln(1+\Delta_{\mathfrak{y}}) = e^{\frac{d}{d\mathfrak{y}}} = \tau_{\mathfrak{y}}$$

we then have that:

$$\int_{\mathbb{Z}_p} \mathfrak{y}^k F(\tau_{\mathfrak{y}}) \approx \int_{\mathbb{Z}_p} \mathfrak{y}^k d\mu_f(\mathfrak{y}) = F^{(k)}(0)$$

which is the $p$-adic analogy of our integration by parts identity:

$$L(\chi,-k) = \frac{1}{\Gamma(-k)} \int_0^\infty t^{-k} F_\chi(t) \frac{dt}{t} = (-1)^k F_\chi^{(k)}(0)$$

Indeed, this analogy is more than just formal. Since we proved the more general statement that:

$$M\{F\}(s) \overset{\text{def}}{=} \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} F(t)\,dt$$

satisfies:

$$M\{F\}(-k) = (-1)^k F^{(k)}(0)$$

we have that

$$\int_{\mathbb{Z}_p} \mathfrak{y}^k d\mu_f(\mathfrak{y}) = \int_{\mathbb{Z}_p} \mathfrak{y}^k F(\tau_{\mathfrak{y}}) = F^{(k)}(0)$$

implies:

$$\int_{\mathbb{Z}_p} \mathfrak{y}^k d\mu_f(\mathfrak{y}) = \int_{\mathbb{Z}_p} \mathfrak{y}^k F(\tau_{\mathfrak{y}}) = (-1)^k M\{F\}(-k)$$

Since $F(t) = f(e^t-1)$, we can write this as:

$$\int_{\mathbb{Z}_p} \mathfrak{y}^k d\mu_f(\mathfrak{y}) = (-1)^k M\left\{f\left(e^t-1\right)\right\}(-k) \tag{40}$$

In summary: multiplication of a measure $d\mu(\mathfrak{y})$ by $\mathfrak{y}^k$ induces a "twisted derivative" $\left((1+x)\frac{d}{dx}\right)^k$ of the **AMT** of $d\mu$; we "unwind" this twisted derivative by replacing $x$ with $e^t - 1$ and writing $\mathscr{A}\{d\mu\}(e^t-1)$ as a power series in $t$. Not only that but equation (?) shows exactly how and where the classical Mellin transform coincides with our $p$-adic integration methods:

(1) Given a function of the form:

$$M\{f\}(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}F(t)\,dt$$

we make the change of variables $x = e^t - 1$ and then define:

$$f(x) = F(\ln(1+x))$$

(2) Taking $f(x)$, we replace $x$ with $\Delta$ to obtain the measure $d\mu_f = f(\Delta)$.

(3) Assuming there aren't any technical difficulties in steps (2) and (3), it then follows that:

$$\int_{\mathbb{Z}_p} \mathfrak{y}^k d\mu_f(\mathfrak{y}) = F^{(k)}(0)\,,\ \forall k \in \mathbb{N}_0$$

Kummer's congruences relating the values of $\zeta(s)$ at negative integers and residues modulo $p$ have thus been generalized to a relation between the values of analytic continuations of Mellin transforms at negative integers and $p$-adic integrals.

## Section IV: Multiply by $p$ - Revisiting Dirichlet Series and the Construction of $p$-Adic $L$-functions

If we want to extend the domain of definition of a function like $\sin x$ to include complex numbers, we simply plug the complex input into the power series $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ and compute the answer. Unfortunately, the same cannot be done for $L$-functions. Consider, for instance, the Riemann-Zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This dirichlet series is convergent for all integer values of $s \geq 2$. Since those integer inputs are dense in $\mathbb{Z}_p$, we would expect them to tell us how to satisfyingly interpolate $\zeta(s)$ to the $p$-adics. However, this cannot be done, for the integers have conspired against us. Since the indices of the series run over all positive integers, it follows that $\frac{1}{p^s}, \frac{1}{p^{2s}}, \frac{1}{p^{3s}}, \ldots$ will all be terms in $\zeta(s)$. For any integers $k \geq 1$ and $s \geq 2$, these terms have $p$-adic magnitude $\left|\frac{1}{p^{ks}}\right|_p = p^{ks}$, which tends to infinity as $k \to \infty$; that is, $\zeta(s)$ contains terms of arbitrarily large $p$-adic magnitude. The same argument dooms any naïve effort to $p$-adically interpolate any $L$-function.

The solution to overcoming this obstacle, however, is as simple as it is old. Following Euler (who himself was following Erastothenes), we write as follows:

$$\begin{aligned}
\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \\
\left(\text{pull out all multiples of } \frac{1}{p^s}\right); \quad &= \underbrace{\left(\frac{1}{p^s} + \frac{1}{(2p)^s} + \frac{1}{(3p)^s} + \cdots\right)}_{\text{all multiples of } p^s} + \text{everything else} \\
&= \sum_{m=1}^{\infty} \frac{1}{(pm)^s} + \sum_{n \in \mathbb{N}_1 : p \nmid n} \frac{1}{n^s} \\
\left(\text{factor out } \frac{1}{p^s}\right); \quad &= \frac{1}{p^s} \underbrace{\sum_{m=1}^{\infty} \frac{1}{m^s}}_{\zeta(s)} + \underbrace{\sum_{n \in \mathbb{N}_1 : p \nmid n} \frac{1}{n^s}}_{\text{call this } \zeta_p^{\times}(s)} \\
&= p^{-s}\zeta(s) + \zeta_p^{\times}(s) \\
&\Updownarrow \\
\left(1 - p^{-s}\right)\zeta(s) &= \zeta_p^{\times}(s)
\end{aligned}$$

If we were to perform this procedure for every prime $p$, we would eventually obtain the famous Euler product for $\zeta(s)$:

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}}$$

where $\mathbb{P}$ is the set of prime numbers in $\mathbb{Z}$ $(2, 3, 5, 7, 11, \ldots)$. However, for our purposes, we need only do this for one $p$ at a time. In doing so, we have obtained the function $\zeta_p^{\times}(s)$, a Dirichlet series whose terms all have magnitude 1 in $\mathbb{Q}_p$. Just as $\zeta(s)$ has a meromorphic continuation to the left-half plane, so too does $\zeta_p^{\times}(s)$.

Now, since we want to work with measures and integral transforms, to proceed, we need to find the function $f_p(x)$ whose Mellin transform is equal to $\zeta_p^{\times}(s)$. Thankfully, this is relatively simple:

$$\begin{array}{rcl}
\zeta_p^{\times}(s) & = & \left(1-p^{-s}\right)\zeta(s) \\
 & = & \zeta(s) - \dfrac{1}{p^s}\zeta(s) \\
 & = & \displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{n^s} - \frac{1}{(pn)^s}\right) \\
 & = & \displaystyle\frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\left(\frac{\Gamma(s)}{n^s} - \frac{\Gamma(s)}{(np)^s}\right) \\
\left(\dfrac{\Gamma(s)}{m^s} = \mathscr{M}\left\{e^{-mt}\right\}(s)\right); & = & \displaystyle\frac{1}{\Gamma(s)}\sum_{n=1}^{\infty}\left(\mathscr{M}\left\{e^{-nt}\right\}(s) - \mathscr{M}\left\{e^{-npt}\right\}(s)\right) \\
 & = & \displaystyle\frac{1}{\Gamma(s)}\mathscr{M}\left\{\sum_{n=1}^{\infty}\left(e^{-nt} - e^{-npt}\right)\right\}(s) \\
 & = & \displaystyle\frac{1}{\Gamma(s)}\mathscr{M}\left\{\frac{e^{-t}}{1-e^{-t}} - \frac{e^{-pt}}{1-e^{-pt}}\right\}(s) \\
\left(\times\dfrac{e^t}{e^t}; \times\dfrac{e^{pt}}{e^{pt}}\right); & = & \displaystyle\frac{1}{\Gamma(s)}\mathscr{M}\left\{\frac{1}{e^t-1} - \frac{1}{e^{pt}-1}\right\}(s)
\end{array}$$

Thus, the associated function (soon to be interpreted as a measure) is $\frac{1}{e^t-1} - \frac{1}{e^{pt}-1}$.

To interpret this function as a $p$-adic measure, we use our analogy, identifying

$e^t$ with the translation operator $\tau = e^D = 1 + \Delta$, observe that we can write:

$$\begin{aligned}
\frac{1}{e^t-1} - \frac{1}{e^{pt}-1} &\cong \frac{1}{\tau - 1} - \frac{1}{\tau^p - 1} \\
&= \frac{1}{(1+\Delta)-1} - \frac{1}{(1+\Delta)^p - 1} \\
&= \frac{1}{\Delta} - \frac{1}{(1+\Delta)^p - 1} \\
\text{(Expand } (1+\Delta)^p\text{)};\quad &= \frac{1}{\Delta} - \frac{1}{\left(1 + \binom{p}{1}\Delta + \cdots + \binom{p}{p}\Delta^p\right) - 1} \\
&= \frac{1}{\Delta} - \frac{1}{p\Delta + \binom{p}{2}\Delta^2 \cdots + \binom{p}{p}\Delta^p} \\
&= \frac{1}{\Delta} - \frac{1}{p\Delta}\frac{1}{1 + \frac{1}{p}\binom{p}{2}\Delta \cdots + \frac{1}{p}\binom{p}{p}\Delta^{p-1}} \\
&= \frac{1}{\Delta}\left(1 - \frac{1}{p}\frac{1}{1 + \Delta\underbrace{\left(\frac{1}{p}\binom{p}{2} + \frac{1}{p}\binom{p}{3}\Delta + \cdots + \frac{1}{p}\binom{p}{p}\Delta^{p-2}\right)}_{\overset{\text{def}}{=} g_p(\Delta)}}\right) \\
&= \frac{1}{\Delta}\left(1 - \frac{1}{p}\frac{1}{1 + \Delta g_p(\Delta)}\right) \\
&= \frac{1}{\Delta}\left(1 - \frac{1}{p}\sum_{n=0}^{\infty}(-\Delta)^n g_p^n(\Delta)\right) \\
&= \frac{1}{\Delta}\left(1 - \frac{1}{p}\left(1 + \sum_{n=1}^{\infty}(-\Delta)^n g_p^n(\Delta)\right)\right) \\
&= \frac{1}{\Delta}\left(1 - \frac{1}{p}\right) + \frac{1}{p}\sum_{n=1}^{\infty}(-\Delta)^{n-1} g_p^n(\Delta)
\end{aligned}$$

But, look: although:

$$g_p(\Delta) = \sum_{k=2}^{p}\binom{p}{k}\Delta^{k-2} = \frac{1}{p}\binom{p}{2} + \frac{1}{p}\binom{p}{3}\Delta + \cdots + \frac{1}{p}\binom{p}{p}\Delta^{p-2}$$

is a polynomial in $\Delta$, note that $\frac{1}{\Delta}\left(1 - \frac{1}{p}\right)$ is not! That is to say, the series associated to this object (obtained by replacing $\Delta$ with $x$) is not a power series, but a *Laurent series*. Thus, our construction still needs some tweaking.

First, note that the $\frac{1}{p}$ came from the expression:

$$\frac{1}{(1+\Delta)^p - 1} = \frac{1}{p\Delta}\frac{1}{1 + \frac{1}{p}\binom{p}{2}\Delta \cdots + \frac{1}{p}\binom{p}{p}\Delta^{p-1}}$$

Thus, we can cancel out that $\frac{1}{p}$ by putting a $p$ on top of $\frac{1}{(1+\Delta)^p-1}$. Writing:

$$f_p(x) \overset{\text{def}}{=} \frac{1}{x} - \frac{p}{(1+x)^p - 1}$$

repeating the above computation gives:

$$f_p(\Delta) = \frac{1}{\Delta}\left(1 - \frac{p}{p}\right) + \frac{p}{p}\sum_{n=1}^{\infty}(-\Delta)^{n-1} g_p^n(\Delta) = \sum_{n=1}^{\infty}(-\Delta)^{n-1} g_p^n(\Delta)$$

which is a power series. But this is still not good enough! Note that the leading term of $g_p(\Delta)$ is $\frac{1}{p}\binom{p}{p}\Delta^{p-2}$, i.e. ,$\frac{1}{p}\Delta^{p-2}$. Remember that we want our measure to be an element of the $K$-Iwasawa algebra—that is, to be a measure whose values (i.e., $\Delta$-series coefficients) are *integers* in $K$. However, the rational number $\frac{1}{p}$ *never* belongs to $\mathcal{O}_K$, indeed, $\frac{1}{p}$ is not even an element of $\mathbb{Z}_p$.

To fix things, we need to step away from using $p$. If we used an integer $a \in \mathbb{Z}$ which was co prime to $p$, and wrote:

$$f_a(x) \overset{\text{def}}{=} \frac{1}{x} - \frac{a}{(1+x)^a - 1} \tag{41}$$

then $f_a(x)$ is the Amice-Mahler transform of the measure:

$$f_a(\Delta) \overset{\text{def}}{=} \frac{1}{\Delta} - \frac{a}{(1+\Delta)^a - 1} = \sum_{n=1}^{\infty}(-\Delta)^{n-1} g_a^n(\Delta) \tag{42}$$

where $g_a(\Delta)$ is the measure defined by the $\Delta$-polynomial:

$$g_a(\Delta) \overset{\text{def}}{=} \sum_{k=2}^{a}\binom{a}{k}\Delta^{k-2} = \frac{1}{a}\binom{a}{2} + \frac{1}{a}\binom{a}{3}\Delta + \cdots + \frac{1}{a}\binom{a}{a}\Delta^{a-2} \tag{43}$$

whose leading term is $\frac{1}{a}\Delta^{a-2}$. Luckily for us, the $\frac{1}{a}$ is no longer a nuisance: since $a$ is co-prime to $p$, $a$ is an invertible element of $\mathbb{Z}_p$, and hence, $\frac{1}{a}$ is an "integer" in $\mathbb{Q}_p$ (i.e., $\frac{1}{a} \in \mathbb{Z}_p$), and thus, in $K$ as well. *More generally*, for any $\mathfrak{a} \in \mathbb{Z}_p^\times$ (i.e., $\mathfrak{a}$ is an invertible element of $\mathbb{Z}_p$), this construction will work, in the sense that the measure:

$$d\mu_{\mathfrak{a}} \overset{\text{def}}{=} f_{\mathfrak{a}}(\Delta) \overset{\text{def}}{=} \frac{1}{\Delta} - \frac{\mathfrak{a}}{(1+\Delta)^{\mathfrak{a}} - 1} = \sum_{n=1}^{\infty}(-\Delta)^{n-1} g_{\mathfrak{a}}^n(\Delta)$$

with:

$$g_{\mathfrak{a}}(\Delta) \overset{\text{def}}{=} \sum_{k=2}^{\mathfrak{a}}\binom{\mathfrak{a}}{k}\Delta^{k-2}$$

it follows that $d\mu_{\mathfrak{a}}$ is a polynomial in $\Delta$ whose coefficients are integers in $\mathbb{Q}_p$ (and hence, in $\mathcal{O}_K$), and thus, that $d\mu_{\mathfrak{a}}$ is indeed an element of the $K$-Iwasawa algebra of $\mathbb{Z}_p$.

Now that we have our measure $d\mu_{\mathfrak{a}}$, let's figure out what it does when we integrate it. First, lets work with $\mathfrak{a} \in \mathbb{Z}$, with $\mathfrak{a}$ co-prime to $p$. Remembering our $x+1=e^t$ analogy from the previous section, we have the formula:

$$\int_{\mathbb{Z}_p} \mathfrak{y}^k d\mu_f(\mathfrak{y}) = (-1)^k M\left\{f\left(e^t-1\right)\right\}(-k)$$

Here, the measure $d\mu_f$ is our measure $d\mu_{\mathfrak{a}}$, and $f$ is $f_{\mathfrak{a}}$. Thus:

$$\begin{array}{rcl}
\displaystyle\int_{\mathbb{Z}_p} \mathfrak{y}^k d\mu_{\mathfrak{a}}(\mathfrak{y}) & = & (-1)^k M\left\{f_{\mathfrak{a}}\left(e^t-1\right)\right\}(-k) \\
\left(f_{\mathfrak{a}}(x) = \dfrac{1}{x} - \dfrac{\mathfrak{a}}{(1+x)^{\mathfrak{a}}-1}\right); & = & (-1)^k \left(\dfrac{1}{\Gamma(s)}\displaystyle\int_0^\infty t^{s-1}\left(\dfrac{1}{e^t-1} - \dfrac{\mathfrak{a}}{e^{\mathfrak{a}t}-1}\right)dt\right)|_{s=-k} \\
 & = & (-1)^k \left(\underbrace{\dfrac{1}{\Gamma(s)}\displaystyle\int_0^\infty \dfrac{t^{s-1}}{e^t-1}dt}_{\zeta(s)} - \dfrac{\mathfrak{a}}{\Gamma(s)}\displaystyle\int_0^\infty \dfrac{t^{s-1}}{e^{\mathfrak{a}t}-1}dt\right)|_{s=-k} \\
\left(\dfrac{1}{e^{\mathfrak{a}t}-1} = \dfrac{e^{-\mathfrak{a}t}}{1-e^{-\mathfrak{a}t}} = \displaystyle\sum_{n=1}^\infty e^{-n\mathfrak{a}t}\right); & = & (-1)^k \left(\zeta(s) - \displaystyle\sum_{n=1}^\infty \dfrac{\mathfrak{a}}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-n\mathfrak{a}t}dt\right)|_{s=-k} \\
 & = & (-1)^k \left(\zeta(s) - \displaystyle\sum_{n=1}^\infty \dfrac{\mathfrak{a}}{\Gamma(s)}\dfrac{\Gamma(s)}{(n\mathfrak{a})^s}\right)|_{s=-k} \\
 & = & (-1)^k \left(\zeta(s) - \dfrac{\mathfrak{a}}{\mathfrak{a}^s}\displaystyle\sum_{n=1}^\infty \dfrac{1}{n^s}\right)|_{s=-k} \\
 & = & (-1)^k\left(1-\mathfrak{a}^{1-s}\right)\zeta(s)|_{s=-k} \\
 & = & (-1)^k\left(1-\mathfrak{a}^{k+1}\right)\zeta(-k)
\end{array}$$

Since the integers are dense in $\mathbb{Z}_p$, the continuity of the expression $(-1)^k\left(1-\mathfrak{a}^{k+1}\right)\zeta(-k)$ with respect to $\mathfrak{a}$ then allows us to write:

$$\int_{\mathbb{Z}_p} \mathfrak{y}^k d\mu_{\mathfrak{a}}(\mathfrak{y}) = (-1)^k\left(1-\mathfrak{a}^{k+1}\right)\zeta(-k),\ \forall \mathfrak{a} \in \mathbb{Z}_p^\times$$

In summary, we have shown:

**Theorem**: For any $\mathfrak{a} \in \mathbb{Z}_p^\times$, define the measures:

$$g_{\mathfrak{a}}(\Delta) \overset{\text{def}}{=} \sum_{k=2}^{\mathfrak{a}} \binom{\mathfrak{a}}{k}\Delta^{k-2} \tag{44}$$

and:

$$d\mu_{\mathfrak{a}} \overset{\text{def}}{=} f_{\mathfrak{a}}(\Delta) \overset{\text{def}}{=} \frac{1}{\Delta} - \frac{\mathfrak{a}}{(1+\Delta)^{\mathfrak{a}}-1} = \sum_{n=1}^\infty (-\Delta)^{n-1} g_{\mathfrak{a}}^n(\Delta) \tag{45}$$

Then:

$$\int_{\mathbb{Z}_p} \mathfrak{y}^n d\mu_{\mathfrak{a}}(\mathfrak{y}) = (-1)^n \left(1 - \mathfrak{a}^{n+1}\right) \zeta(-n), \ \forall n \in \mathbb{N}_0, \ \forall \mathfrak{a} \in \mathbb{Z}_p^{\times} \tag{46}$$

Thus, $d\mu_{\mathfrak{a}}$ is a $p$-adic measure that produces the non-negative negative values of the Riemann Zeta function. *However*, this measure depends on $\mathfrak{a} \in \mathbb{Z}_p^{\times}$.

## Intermezzo: Haar measures, Harmonic Analysis, and the Idèles

So far, we have been working toward the goal of constructing a measure on $\mathbb{Z}_p$ that mimics the way $L$-functions can be written as the Dirichlet transform of rational functions of $e^t$. $L$-functions, compared to $\zeta(s)$, can be thought of as a variant of $\zeta(s)$ obtained by "twisting" it with a dirichlet character. Indeed, this is explicitly the case in our integral transform formula:

$$L(\chi, s) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left( \sum_{n=1}^{\infty} \chi(n) e^{-nt} \right) dt$$

with $L(1, s) = \zeta(s)$ (where 1 is the trivial character $\chi(n) = 1$ for all $n$). Thus, it stands to reason that, in going from the measure $d\mu_{\mathfrak{a}}$ (which detects values of $\zeta(s) = L(1, s)$) to something which detects values of $L(\chi, s)$ (for any dirichlet character $\chi$), we will probably have to end up sticking a $\chi$ inside our $p$-adic integrals. However, this is where things get complicated.

To begin with, since we are using measures to define integration on $\mathbb{Z}_p$, we must keep in mind that, as a locally compact (in fact, completely compact) abelian group, $\mathbb{Z}_p$ possesses a natural measure—its **Haar measure**, $d\eta$, normalized so that $\int_{\mathbb{Z}_p} d\eta = 1$, and hence, so that $\int_{p^n\mathbb{Z}_p} d\eta = \frac{1}{p^n}$. Since $\frac{1}{p^n} \in \mathbb{Q}_p \backslash \mathbb{Z}_p$ this haar measure, while not an element of the Iwasawa algebra, is nonetheless an element of $\left(C^0(\mathbb{Z}_p, K)\right)^*$ (since $\mathbb{Q}_p \subseteq K$). Haar measures are of stupendous importance because they are the foundation for harmonic analysis on locally compact abelian groups—for instance, they are how we define the Fourier transform of a function in $C^0(\mathbb{Z}_p, \mathbb{C})$, among other things. The neat thing about haar measures is that, once you know about them, you see them everywhere. Indeed, we mentioned earlier that the mellin transform can be viewed as integration with respect to the Haar measure $\frac{dt}{t}$:

$$\mathscr{M}\{f\}(s) = \int_0^{\infty} t^{s-1} f(t)\, dt = \int_0^{\infty} t^s f(t) \underbrace{\frac{dt}{t}}_{\text{haar}}$$

where, as can be shown, $\frac{dt}{t}$ is the Haar measure for $(\mathbb{R}^+, \times)$, the group of positive real numbers under the multiplication operation. Thus, we can just as well write the mellin transform as:

$$\mathscr{M}\{f\}(s) = \int_{(\mathbb{R}^+, \times)} t^s f(t)\, d\eta_{(\mathbb{R}^+, \times)}(t)$$

A rule of thumb in harmonic analysis is that, when dealing with the Fourier-Mellin-Laplace family of integral transforms on a locally compact abelian group (LCAG) $G$, the transforms tend to have a formula of the form:

$$f(x) \mapsto \int_G \xi(x) f(x)\, d\eta_G(x) \tag{47}$$

where $\xi$ is any group homomorphism ("character") from $G$ to the multiplicative group of the unit circle, the multiplicative group of non-zero complex numbers, or the additive group complex numbers (depending on the context). In harmonic analysis, the transform $\int_G \xi(x) f(x) \, d\eta_G(x)$ is thought of as a function of $\xi$; the set $\hat{G}$ of all such characters (usually to the unit circle) is a group—the **Pontryagin dual** of $G$. Most of the time, the evaluation of $\xi$ at $x$ can be realized as a kind of inner product, and is often written as such:

$$\mathscr{F}\{f\}(\xi) = \int_G \langle \xi, x \rangle f(x) \, d\eta_G(x), \ \forall \xi \in \hat{G} \tag{48}$$

where $\hat{G}$ is the group of group homomorphisms $\xi : G \to \partial\mathbb{D}$ and $\langle \xi, x \rangle$ denotes the evaluation of $\xi \in \hat{G}$ at $x \in G$. This is the generalization of the Fourier transform to a LCAG. In the case of the classical Fourier transform on $\mathbb{R}$:

$$\langle \xi, x \rangle = e^{-2\pi i \xi x}$$

because every group homomorphism from $\mathbb{R}$ to $\partial\mathbb{D}$ can be written as a function of the form $x \mapsto e^{-2\pi i \xi x}$ for some $\xi \in \mathbb{R}^+$.

Now, turning back to the mellin transform:

$$\mathscr{M}\{f\}(s) = \int_{(\mathbb{R}^+, \times)} t^s f(t) \, d\eta_{(\mathbb{R}^+, \times)}(t)$$

once can easily show that every group homomorphism from $(\mathbb{R}^+, \times)$ to $(\mathbb{C} \backslash \{0\}, \times)$ is of the form $t \mapsto t^s$ for some $s \in \mathbb{C} \backslash \{0\}$. That is, we can *identify* the group of all such group homomorphisms with the group $(\mathbb{C} \backslash \{0\}, \times)$; every such homomorphism is given by $t \mapsto t^s$ for some *unique* $s \in \mathbb{C} \backslash \{0\}$. Thus, the anatomy of the Mellin transform can be expressed as:

$$\mathscr{M}\{\text{function}\}(\text{mult. character}) = \int_{\text{mult. group}} (\text{mult. character}) \times (\text{function}) \times (\text{haar measure})$$

However, turning back to our $p$-adic integral:

$$\int_{\mathbb{Z}_p} \mathfrak{y}^n d\mu_{\mathfrak{a}}(\mathfrak{y}) = (-1)^n \left(1 - \mathfrak{a}^{n+1}\right) \zeta(-n), \ \forall \mathfrak{a} \in \mathbb{Z}_p^\times$$

note that the domain of integration, $\mathbb{Z}_p$, is an *additive* group, rather than a multiplicative group! Thus, if we truly wish to generalize $L$-functions' Mellin transform formulae, we need to change the domain of integration from the additive group $(\mathbb{Z}_p, +)$ to the multiplicative group $\left(\mathbb{Z}_p^\times, \times\right)$.

Although this detail might seem minor at first, it actually goes quite deep.

Returning to our formula for $L(\chi, s)$, observe that we can write:

$$\begin{aligned} L(\chi, s) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n=1}^{\infty} \chi(n) e^{-nt} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \left( \sum_{n=1}^{\infty} t^s \chi(n) e^{-nt} \right) \frac{dt}{t} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \left( \sum_{n=1}^{\infty} (s, \chi)(n, t) e^{-nt} \right) \frac{dt}{t} \end{aligned}$$

where the map $(s, \chi) : \mathbb{Z} \times \mathbb{R}^+ \to \mathbb{C}^\times$ is defined by:

$$(s, \chi)(t, n) \overset{\text{def}}{=} t^s \chi(n) \tag{49}$$

Note that for any co-prime integers $m, n$ and any positive real numbers $t, \tau$:

$$(s, \chi)(t\tau, mn) = (t\tau)^s \chi(mn) = t^s \chi(m) \times \tau^s \chi(n) = (s, \chi)(t, m) \times (s, \chi)(\tau, n)$$

which is to say, $(s, \chi)$ has a multiplicative group homomorphism property in both of its inputs. In that sense, we can think of $(s, \chi)$ as a kind of **composite character**:

$$\kappa_{s,\chi}(n, t) \overset{\text{def}}{=} (s, \chi)(t, n) = t^s \chi(n) \tag{50}$$

Then, we can write $L$ as a function of $\kappa_{s,\chi}$:

$$L(\kappa_{s,\chi}) = \frac{1}{\Gamma(s)} \int_0^\infty \left( \sum_{n=1}^{\infty} \kappa_{s,\chi}(t, n) e^{-nt} \right) \frac{dt}{t}$$

This then shows us what an $L$-function really is: a function of composite multiplicative characters:

$$\kappa_{s,\chi} : \mathbb{R}^+ \times \prod_{N=1}^{\infty} (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$$

where, recall, for every dirichlet character $\chi$, there is a positive integer $N$ (the **conductor** of $\chi$) so that $\chi$ is a multiplicative group homomorphism from $(\mathbb{Z}/N\mathbb{Z})^\times$ to the unit circle $\partial\mathbb{D}$. By the **Chinese Remainder Theorem** (CRT), we can re-write this as:

$$\kappa_{s,\chi} : \mathbb{R}^+ \times \prod_{\ell \in \mathbb{P}} (\mathbb{Z}/\ell\mathbb{Z})^\times \to \mathbb{C}^\times \tag{51}$$

where $\mathbb{P}$ is the set of prime numbers.

We can even extend this to the $\ell$-adics!

**Definition**: In the case where $\chi$ has a conductor of $\ell^n$ for some prime $\ell$ and some $n \in \mathbb{N}_1$, we can extend $\chi$ from $(\mathbb{Z}/\ell\mathbb{Z})^\times$ to $\mathbb{Z}_\ell^\times$ by defining:

$$\chi(\mathfrak{a}) \overset{\text{def}}{=} \chi\left(\ell^n \left\{\ell^{-n}\mathfrak{a}\right\}_\ell\right) \tag{52}$$

where $\{\cdot\} : \mathbb{Q}_\ell \to \mathbb{Q}$ is the **$\ell$-adic fractional part**, defined by:

$$\left\{ \sum_{m=-m_0}^{\infty} c_m \ell^m \right\} \overset{\text{def}}{=} \sum_{m=-m_0}^{-1} c_m \ell^m$$

*Remark*: We will write $\{x\}$ to denote the fractional part of an $\ell$-adic number $x \in \mathbb{Q}_\ell$ for any $\ell \in \mathbb{N}_2$. This definition then extends to $\chi$s of arbitrary conductor by way of the **CRT**. Consequently, we can think of our composite characters as multiplicative group homomorphisms:

$$\kappa_{s,\chi} : \mathbb{R}^+ \times \prod_{\ell \in \mathbb{P}} \mathbb{Z}_\ell^\times \to \mathbb{C}^\times$$

where we give the domain of these characters the product topology. Equipped with the product topology, the $\mathbb{R}^+ \times \prod_{\ell \in \mathbb{P}} \mathbb{Z}_\ell^\times$ are the **idèles of $\mathbb{Q}$**.

## Section V: To the Finish Line - Removing the Dependence on $\mathfrak{a}$, Gauss Sums, Landau Characters, and the Kubata-Leopoldt $p$-Adic Zeta & $L$-functions

To step down from $\mathbb{Z}_p$ to $\mathbb{Z}_p^\times$ in our formula:

$$\int_{\mathbb{Z}_p} \mathfrak{y}^n d\mu_{\mathfrak{a}}(\mathfrak{y}) = (-1)^n \left(1 - \mathfrak{a}^{n+1}\right) \zeta(-n), \ \forall n \in \mathbb{N}_0, \ \forall \mathfrak{a} \in \mathbb{Z}_p^\times$$

we need to examine the effects of **restricting** measures to compact open subsets of $\mathbb{Z}_p$.

**Definition**: Given a compact-open subset $U \subseteq \mathbb{Z}_p$ and a measure $d\mu \in \left(C^0(\mathbb{Z}_p, K)\right)^*$, we write $d\mu_{|U}$ to denote the **restriction of $d\mu$ to $U$**, the measure defined by the property:

$$\int_{\mathbb{Z}_p} f(\mathfrak{y})\, d\mu_{|U}(\mathfrak{y}) \overset{\text{def}}{=} \int_{\mathbb{Z}_p} f(\mathfrak{y})\, \mathbf{1}_U(\mathfrak{y})\, d\mu(\mathfrak{y}), \ \forall f(\mathfrak{y}) \in C^0(\mathbb{Z}_p, K) \tag{53}$$

**Formula**: Using the fact that $\mathbb{Z}_p$ contains $p^n$th roots of unity for all $n \in \mathbb{N}_1$ (and that any such root of unity $\xi$ satisfies $|\xi - 1|_p < 1$), we can write out an explicit formula for compact open subsets $U \subseteq \mathbb{Z}_p$ of the form $U = x_0 + p^n\mathbb{Z}_p$, where $x_0 \in \mathbb{Z}_p$ and $n \in \mathbb{N}_1$:

$$\mathbf{1}_{x_0 + p^n\mathbb{Z}_p}(x) = \frac{1}{p^n} \sum_{\xi \in \mathbb{Z}_p : \xi^{p^n} = 1} \xi^{x - x_0} \tag{54}$$

Now, using the Amice-Mahler Transform, given a measure $d\mu$, let's see what $x$ power series corresponds to $d\mu_{|x_0 + p^n\mathbb{Z}_p}$. But first, a simple computation:

**Proposition**: For any $\xi \in \mathbb{Z}_p$ with $|\xi - 1|_p < 1$ and any $d\mu \in \left(C^0(\mathbb{Z}_p, K)\right)^*$:

$$\mathscr{A}\{\xi^{\mathfrak{y}} d\mu\}(x) = \mathscr{A}\{d\mu\}((1 + x)\xi - 1) \tag{55}$$

Proof:

$$\begin{aligned}
\mathscr{A}\{\xi^{\mathfrak{y}} d\mu\}(x) &= \int_{\mathbb{Z}_p} \xi^{\mathfrak{y}} (1 + x)^{\mathfrak{y}} d\mu(\mathfrak{y}) \\
&= \int_{\mathbb{Z}_p} ((1 + x)\xi)^{\mathfrak{y}} d\mu(\mathfrak{y}) \\
&= \int_{\mathbb{Z}_p} (1 + ((1 + x)\xi - 1))^{\mathfrak{y}} d\mu(\mathfrak{y}) \\
&= \mathscr{A}\{d\mu\}((1 + x)\xi - 1)
\end{aligned}$$

The inequality $|\xi - 1|_p < 1$ is required so as to guarantee that

$$\xi^{\mathfrak{y}} = (1 + (\xi - 1))^{\mathfrak{y}} = \sum_{n=0}^{\infty} \binom{\mathfrak{y}}{n} (\xi - 1)^n$$

converges, so as to ensure that the identity:

$$\xi^{\mathfrak{y}}\left(1+x\right)^{\mathfrak{y}}=\left(\xi\left(1+x\right)\right)^{\mathfrak{y}}$$

is true.

Q.E.D.

**Corollary**: For any $d\mu\in\left(C^{0}\left(\mathbb{Z}_{p},K\right)\right)^{*}$, any $\mathfrak{y}_{0}\in\mathbb{Z}_{p}$ and any $n\in\mathbb{N}_{1}$:

$$\mathscr{A}\left\{ d\mu_{|\mathfrak{y}_{0}+p^{n}\mathbb{Z}_{p}}\right\} \left(x\right)=\frac{1}{p^{n}}\sum_{\xi\in\mathbb{Z}_{p}:\xi^{p^{n}}=1}\xi^{-\mathfrak{y}_{0}}\mathscr{A}\left\{ d\mu\right\} \left(\left(1+x\right)\xi-1\right)\tag{56}$$

Proof:

$$\begin{aligned}
\mathscr{A}\left\{ d\mu_{|\mathfrak{y}_{0}+p^{n}\mathbb{Z}_{p}}\right\} \left(x\right) & = \int_{\mathbb{Z}_{p}}\left(1+x\right)^{\mathfrak{y}}\mathbf{1}_{\mathfrak{y}_{0}+p^{n}\mathbb{Z}_{p}}\left(\mathfrak{y}\right)d\mu\left(\mathfrak{y}\right)\\
\left(\text{Formula for }\mathbf{1}_{\mathfrak{y}_{0}+p^{n}\mathbb{Z}_{p}}\left(\mathfrak{y}\right)\right); & = \int_{\mathbb{Z}_{p}}\left(1+x\right)^{\mathfrak{y}}\left(\frac{1}{p^{n}}\sum_{\xi\in\mathbb{Z}_{p}:\xi^{p^{n}}=1}\xi^{\mathfrak{y}-\mathfrak{y}_{0}}\right)d\mu\left(\mathfrak{y}\right)\\
\left(\xi^{p^{n}}=1\Rightarrow\left|\xi-1\right|_{p}<1\right); & = \frac{1}{p^{n}}\sum_{\xi\in\mathbb{Z}_{p}:\xi^{p^{n}}=1}\xi^{-\mathfrak{y}_{0}}\int_{\mathbb{Z}_{p}}\left(1+x\right)^{\mathfrak{y}}\xi^{\mathfrak{y}}d\mu\left(\mathfrak{y}\right)\\
 & = \frac{1}{p^{n}}\sum_{\xi\in\mathbb{Z}_{p}:\xi^{p^{n}}=1}\xi^{-\mathfrak{y}_{0}}\mathscr{A}\left\{ \xi^{\mathfrak{y}}d\mu\right\} \left(x\right)\\
 & = \frac{1}{p^{n}}\sum_{\xi\in\mathbb{Z}_{p}:\xi^{p^{n}}=1}\xi^{-\mathfrak{y}_{0}}\mathscr{A}\left\{ d\mu\right\} \left(\left(1+x\right)\xi-1\right)
\end{aligned}$$

$$\mathbf{1}_{x_{0}+p^{n}\mathbb{Z}_{p}}\left(x\right)=\frac{1}{p^{n}}\sum_{\xi\in\mathbb{Z}_{p}:\xi^{p^{n}}=1}\xi^{x-x_{0}}$$

**Multiplicative Restriction Formula**: Since $\mathbb{Z}_{p}$ is the union of the disjoint sets $\mathbb{Z}_{p}^{\times}$ and $p\mathbb{Z}_{p}$, it follows that:

$$\mathbf{1}_{\mathbb{Z}_{p}}\left(\mathfrak{y}\right)=\mathbf{1}_{\mathbb{Z}_{p}^{\times}}\left(\mathfrak{y}\right)+\mathbf{1}_{p\mathbb{Z}_{p}}\left(\mathfrak{y}\right)$$

and thus, that:

$$\mathbf{1}_{\mathbb{Z}_{p}^{\times}}\left(\mathfrak{y}\right)=\mathbf{1}_{\mathbb{Z}_{p}}\left(\mathfrak{y}\right)-\mathbf{1}_{p\mathbb{Z}_{p}}\left(\mathfrak{y}\right)$$

Hence:

$$d\mu_{|\mathbb{Z}_{p}^{\times}}=d\mu-d\mu_{|p\mathbb{Z}_{p}}\tag{57}$$

and so, by the previous **Proposition**:

$$\begin{aligned}
\mathscr{A}\left\{ d\mu_{|\mathbb{Z}_{p}^{\times}}\right\} \left(x\right) & = \mathscr{A}\left\{ d\mu\right\} \left(x\right)-\mathscr{A}\left\{ d\mu_{|p\mathbb{Z}_{p}}\right\} \left(x\right)\\
 & = \mathscr{A}\left\{ d\mu\right\} \left(x\right)-\frac{1}{p}\sum_{\xi\in\mathbb{Z}_{p}:\xi^{p}=1}\mathscr{A}\left\{ d\mu\right\} \left(\left(1+x\right)\xi-1\right)
\end{aligned}$$

Thus:

$$\mathscr{A}\left\{d\mu_{|\mathbb{Z}_p^\times}\right\}(x) = \mathscr{A}\left\{d\mu\right\}(x) - \frac{1}{p}\sum_{\xi\in\mathbb{Z}_p:\xi^p=1}\mathscr{A}\left\{d\mu\right\}\left((1+x)\,\xi-1\right) \tag{58}$$

With this in hand, we then write:

$$\begin{aligned}
\int_{\mathbb{Z}_p^\times}\mathfrak{y}^k d\mu_{\mathfrak{a}}(\mathfrak{y}) &= \int_{\mathbb{Z}_p}\mathfrak{y}^k\mathbf{1}_{\mathbb{Z}_p^\times}(\mathfrak{y})\,d\mu_{\mathfrak{a}}(\mathfrak{y}) \\
&= \int_{\mathbb{Z}_p}\mathfrak{y}^k\left(\mathbf{1}_{\mathbb{Z}_p}(\mathfrak{y})-\mathbf{1}_{p\mathbb{Z}_p}(\mathfrak{y})\right)d\mu_{\mathfrak{a}}(\mathfrak{y}) \\
&= \int_{\mathbb{Z}_p}\mathfrak{y}^k\mathbf{1}_{\mathbb{Z}_p}(\mathfrak{y})\,d\mu_{\mathfrak{a}}(\mathfrak{y}) - \int_{\mathbb{Z}_p}\mathfrak{y}^k\mathbf{1}_{p\mathbb{Z}_p}(\mathfrak{y})\,d\mu_{\mathfrak{a}}(\mathfrak{y}) \\
&= \int_{\mathbb{Z}_p}\mathfrak{y}^k d\mu_{\mathfrak{a}}(\mathfrak{y}) - \int_{\mathbb{Z}_p}(p\mathfrak{y})^k d\mu_{\mathfrak{a}}(\mathfrak{y}) \\
&= \int_{\mathbb{Z}_p}\mathfrak{y}^k d\mu_{\mathfrak{a}}(\mathfrak{y}) - p^k\int_{\mathbb{Z}_p}\mathfrak{y}^k d\mu_{\mathfrak{a}}(\mathfrak{y}) \\
&= \left(1-p^k\right)\int_{\mathbb{Z}_p}\mathfrak{y}^k d\mu_{\mathfrak{a}}(\mathfrak{y}) \\
\left(\int_{\mathbb{Z}_p}\mathfrak{y}^k d\mu_{\mathfrak{a}}(\mathfrak{y}) = (-1)^k\left(1-\mathfrak{a}^{k+1}\right)\zeta(-k)\right);\quad &= (-1)^k\left(1-p^k\right)\left(1-\mathfrak{a}^{k+1}\right)\zeta(-k)
\end{aligned}$$

This proves:

$$\int_{\mathbb{Z}_p^\times}\mathfrak{y}^k d\mu_{\mathfrak{a}}(\mathfrak{y}) = (-1)^k\left(1-p^k\right)\left(1-\mathfrak{a}^{k+1}\right)\zeta(-k)\,,\ \forall k\in\mathbb{N}_0\ \forall\mathfrak{a}\in\mathbb{Z}_p^\times \tag{59}$$

where:

$$d\mu_{\mathfrak{a}} \overset{\text{def}}{=} \frac{1}{\Delta} - \frac{\mathfrak{a}}{(1+\Delta)^{\mathfrak{a}}-1} = \sum_{n=1}^{\infty}(-\Delta)^{n-1}g_{\mathfrak{a}}^n(\Delta)$$

where:

$$g_{\mathfrak{a}}(\Delta) \overset{\text{def}}{=} \sum_{k=2}^{\mathfrak{a}}\binom{\mathfrak{a}}{k}\Delta^{k-2}$$

With that done, we can finally proceed to remove the dependence on $\mathfrak{a}$. The idea is to expand our definition of measures to include *quotients* of measures—objects called **pseudo-measures**.

**Definition**: We write $Q_K(\mathbb{Z}_p)$ to denote **the field of fractions of** $\Lambda_K(\mathbb{Z}_p)$.. A **pseudo-measure** on $C^0(\mathbb{Z}_p,\mathcal{O}_K)$ is an element $d\lambda\in Q_K(\mathbb{Z}_p)$ so that $(\mathfrak{y}-1)\,d\lambda(\mathfrak{y})$ is an element of $\Lambda_K(\mathbb{Z}_p)$ for all $\mathfrak{y}\in\mathbb{Z}_p$. We denote the output of a pseudo-measure $d\lambda=\frac{d\mu}{d\nu}$ evaluated on a function $f\in C^0(\mathbb{Z}_p,\mathcal{O}_K)$ as

$\int_{\mathbb{Z}_p} f(\mathfrak{y})\, d\lambda(\mathfrak{y})$, and we define this quantity as:

$$\int_{\mathbb{Z}_p} f(\mathfrak{y})\, d\lambda(\mathfrak{y}) \overset{\text{def}}{=} \frac{\int_{\mathbb{Z}_p} f(\mathfrak{y})\, d\mu(\mathfrak{y})}{\int_{\mathbb{Z}_p} f(\mathfrak{y})\, d\nu(\mathfrak{y})}$$

(Note: the authors of my source do not define this action; I am merely guessing that this is what it is).

**Definition:** For any $\mathfrak{a} \in \mathbb{Z}_p$, we define the **(Dirac) delta measure at $\mathfrak{a}$** as the functional $\delta_{\mathfrak{a}} \in \left(C^0(\mathbb{Z}_p, K)\right)^*$ satisfying the property:

$$\int_{\mathbb{Z}_p} f(\mathfrak{y})\, \delta_{\mathfrak{a}}(\mathfrak{y}) \overset{\text{def}}{=} f(\mathfrak{a}),\ \forall f \in C^0(\mathbb{Z}_p, K)$$

**Observations**: For all $\mathfrak{a} \in \mathbb{Z}_p$, all $f \in C^0(\mathbb{Z}_p, K)$, and all $k \in \mathbb{N}_0$
I. $\mathscr{A}\{\delta_{\mathfrak{a}}\}(x) = \int_{\mathbb{Z}_p} (1+x)^{\mathfrak{y}}\, \delta_{\mathfrak{a}}(\mathfrak{y}) = (1+x)^{\mathfrak{a}}$
II. $\int_{\mathbb{Z}_p} f(\mathfrak{y})\left(\delta_{\mathfrak{a}}(\mathfrak{y}) - \delta_1(\mathfrak{y})\right) = f(\mathfrak{a}) - f(1)$
III. $\int_{\mathbb{Z}_p} \mathfrak{y}^k \left(\delta_{\mathfrak{a}}(\mathfrak{y}) - \delta_1(\mathfrak{y})\right) = \mathfrak{a}^k - 1$

**Definition**: The **Kubata-Leopoldt *p*-adic *L*-function** (or ***p*-adic Zeta function**), denoted $\zeta_p$, is the pseudo-measure defined by $\zeta_p(\mathfrak{y}) \overset{\text{def}}{=} \frac{d\mu_{\mathfrak{a}}(\mathfrak{y})}{\mathfrak{y}(\delta_{\mathfrak{a}}(\mathfrak{y}) - \delta_1(\mathfrak{y}))}$ for any $\mathfrak{a} \in \mathbb{Z}_p^{\times}$. This definition is independent of the choice of $\mathfrak{a}$. (To see the independence, simply observe that (by evaluation against $\mathfrak{y}^k$ for any $k \in \mathbb{N}_0$):

$$\mathfrak{y}\left(\delta_{\mathfrak{b}}(\mathfrak{y}) - \delta_1(\mathfrak{y})\right) d\mu_{\mathfrak{a}}(\mathfrak{y}) = \mathfrak{y}\left(\delta_{\mathfrak{a}}(\mathfrak{y}) - \delta_1(\mathfrak{y})\right) d\mu_{\mathfrak{b}}(\mathfrak{y})$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathbb{Z}_p^{\times}$.)

By construction:

$$\begin{aligned}
\int_{\mathbb{Z}_p^{\times}} \mathfrak{y}^k \zeta_p(\mathfrak{y}) &= \frac{\int_{\mathbb{Z}_p^{\times}} \mathfrak{y}^k d\mu_{\mathfrak{a}}(\mathfrak{y})}{\int_{\mathbb{Z}_p^{\times}} \mathfrak{y}^k \mathfrak{y}\left(\delta_{\mathfrak{a}}(\mathfrak{y}) - \delta_1(\mathfrak{y})\right)} \\
&= \frac{(-1)^k \left(1 - p^k\right)\left(1 - \mathfrak{a}^{k+1}\right)\zeta(-k)}{\mathfrak{a}^{k+1} - 1} \\
&= (-1)^{k+1}\left(1 - p^k\right)\zeta(-k) \\
&= \begin{cases} -\left(1 - p^{2j}\right)\zeta(-2j) & \text{if } k = 2j \\ \left(1 - p^{2j+1}\right)\zeta(-(2j+1)) & \text{if } k = 2j+1 \end{cases} \\
(\zeta(-\text{even}) = 0);\quad &= \begin{cases} 0 & \text{if } k = 2j \\ \left(1 - p^{2j+1}\right)\zeta(-(2j+1)) & \text{if } k = 2j+1 \end{cases} \\
(\zeta(-\text{even}) = 0);\quad &= \begin{cases} \left(1 - p^{2j}\right)\zeta(-2j) & \text{if } k = 2j \\ \left(1 - p^{2j+1}\right)\zeta(-(2j+1)) & \text{if } k = 2j+1 \end{cases} \\
&= \left(1 - p^k\right)\zeta(-k)
\end{aligned}$$

To deal with the case of a general $L$-function $L(\chi, s)$, we return to our Mellin transform integral formulae, this time with our $e^t = x+1 \sim e^D = \Delta+1$ analogy in mind. Fix a prime $p$, and let $\chi$ be a Dirichlet character of mod $\beta$ for some $\beta \in \mathbb{N}_1$, we can exploit $\chi$'s multiplicative property ($\chi(mn) = \chi(m)\chi(n)$ for all co-prime integers $m$ and $n$) to repeat the same decomposition for an arbitrary $L$-function, albeit with a slight modification to deal with the $N$-periodicity of $\chi$:

$$\begin{aligned}
L(\chi, s) &= \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} \\
\left(\text{split } \mathbb{N}_1 \bmod \beta : \mathbb{N}_1 = \bigcup_{k=1}^{\beta} (\beta\mathbb{N}_0 + k)\right); \quad &= \sum_{k=1}^{\beta}\sum_{n=0}^{\infty} \frac{\chi(\beta n + k)}{(\beta n + k)^s} \\
(\chi \text{ is } \beta - \text{periodic}); \quad &= \sum_{k=1}^{\beta}\sum_{n=0}^{\infty} \frac{\chi(k)}{(\beta n + k)^s} \\
&= \sum_{k=1}^{\beta} \frac{\chi(k)}{\beta^s} \sum_{n=0}^{\infty} \frac{1}{\left(n + \frac{k}{\beta}\right)^s} \\
\left(\zeta(s,q) \overset{\text{def}}{=} \sum_{n=0}^{\infty} \frac{1}{(n+q)^s},\ \operatorname{Re}(q) > 0\right); \quad &= \beta^{-s} \sum_{k=1}^{\beta} \chi(k)\, \zeta\left(s, \frac{k}{\beta}\right)
\end{aligned}$$

where $\zeta(s,q)$ is the **Hurwitz Zeta Function (HZF)**. Now, writing:

$$\begin{aligned}
\zeta(s,q) &= \sum_{n=0}^{\infty} \frac{1}{(n+q)^s} \\
&= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{\Gamma(s)}{(n+q)^s} \\
\left(\mathscr{M}\left\{e^{-ax}\right\}(s) = \frac{\Gamma(s)}{a^s}\right); \quad &= \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \mathscr{M}\left\{e^{-(n+q)x}\right\}(s) \\
&= \frac{1}{\Gamma(s)} \mathscr{M}\left\{e^{-qx} \sum_{n=0}^{\infty} e^{-nx}\right\}(s) \\
&= \frac{1}{\Gamma(s)} \mathscr{M}\left\{\frac{e^{-qx}}{1-e^{-x}}\right\}(s) \\
&= \frac{1}{\Gamma(s)} \int_0^{\infty} x^{s-1} \frac{e^{-qx}}{1-e^{-x}} dx
\end{aligned}$$

we obtain a Mellin transform formula for the HZF. As such, for a dirichlet character $\chi$ modulo $\beta$, we have the formula:

$$\begin{aligned}
L(\chi, s) &= \sum_{k=1}^{\beta} \chi(k) \zeta\left(s, \frac{k}{\beta}\right) \\
&= \sum_{k=1}^{\beta} \chi(k) \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \frac{e^{-\frac{k}{\beta}u}}{1-e^{-u}} du \\
\left(\times \frac{e^u}{e^u}\right); \quad &= \frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \sum_{k=1}^{\beta} \frac{\chi(k) e^{\left(1-\frac{k}{\beta}\right)u}}{e^u - 1} du \\
(u = \beta t); \quad &= \frac{1}{\Gamma(s)} \int_0^\infty (\beta t)^{s-1} \sum_{k=1}^{\beta} \frac{\chi(k) e^{\left(1-\frac{k}{\beta}\right)\beta t}}{e^{\beta t} - 1} \beta dt \\
&= \frac{\beta^s}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{k=1}^{\beta} \frac{\chi(k) e^{(\beta-k)t}}{e^{\beta t} - 1} dt \\
(j = \beta - k); \quad &= \frac{\beta^s}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{j=0}^{\beta-1} \frac{\chi(j+\beta) e^{jt}}{e^{\beta t} - 1} dt \\
(\chi(j+\beta) = \chi(j)); \quad &= \frac{\beta^s}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{k=0}^{\beta-1} \frac{\chi(k) e^{kt}}{e^{\beta t} - 1} dt
\end{aligned}$$

So:

$$L(\chi, s) = \frac{\beta^s}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{k=0}^{\beta-1} \frac{\chi(k) e^{kt}}{e^{\beta t} - 1} dt$$

for any dirichlet character $\chi$ of modulus $\beta$.

Now, like before, we are going to want to exploit our $e^t = x+1 \sim e^D = \Delta+1$ analogy. However, the rational function $\frac{e^{kt}}{e^{\beta t}-1}$ is going to be a bit messy to work with, because, after the change of variables to $x$, the measure associated to is is:

$$\frac{(\Delta+1)^k}{(\Delta+1)^\beta - 1}$$

which means having to expand $(\Delta+1)^\beta$ using the binomial formula. Fortunately for us, however, there is a trick we can employ.

**Definition**: For any dirichlet character $\chi$, write $|\chi|$ to denote the modulus of $\chi$. Then, the **Gauss sum** of $\chi$ is the quantity:

$$G(\chi) \overset{\text{def}}{=} \sum_{n=0}^{|\chi|-1} \chi(n) e^{-\frac{2n\pi i}{|\chi|}}$$

Note that $G(\chi)$ is merely the **finite Fourier transform (mod $|\chi|$)** of the sequence $\{\chi(n)\}_{0 \leq n \leq |\chi|-1}$:

$$\hat{\chi}(k) \overset{\text{def}}{=} \mathscr{F}_{|\chi|}\{\chi\}(k) \overset{\text{def}}{=} \sum_{n=0}^{|\chi|-1} \chi(n)\, e^{-\frac{2nk\pi i}{|\chi|}}$$

evaluated at $k = 1$. Since $\chi(n) \in \partial\mathbb{D}$, we see that:

$$\chi(n)\,\overline{\chi(n)} = 1,\ \forall n \in (\mathbb{Z}/|\chi|\,\mathbb{Z})^{\times}$$

and hence, $\chi^{-1} = \overline{\chi}$. Consequently:

$$\overline{\hat{\chi}(k)} = \overline{\sum_{n=0}^{|\chi|-1} \chi(n)\, e^{-\frac{2nk\pi i}{|\chi|}}} = \sum_{n=0}^{|\chi|-1} \overline{\chi(n)} e^{\frac{2nk\pi i}{|\chi|}} = \sum_{n=0}^{|\chi|-1} \chi^{-1}(n)\, e^{-\frac{2nk\pi i}{|\chi|}} = \hat{\overline{\chi}}(k)$$

That is, the maps "take Fourier transform of $\chi$" and "send $\chi$ to $\chi^{-1}$" commute with one another. This is extremely useful.

First, the fact that $\chi$ is non-vanishing on $(\mathbb{Z}/|\chi|\,\mathbb{Z})^{\times}$, and, that the map $n \to nk$ is a bijection of $(\mathbb{Z}/|\chi|\,\mathbb{Z})^{\times}$ for all $k \in (\mathbb{Z}/|\chi|\,\mathbb{Z})^{\times}$ allow us to write:

$$\begin{aligned}
\hat{\chi}(k) &= \sum_{n \in (\mathbb{Z}/|\chi|\mathbb{Z})^{\times}} \chi(n)\, e^{-\frac{2nk\pi i}{|\chi|}} \\
(m = nk);\ &= \sum_{m \in [k]_{\chi}^{-1}(\mathbb{Z}/|\chi|\mathbb{Z})^{\times}} \chi\left(m\,[k]_{\chi}^{-1}\right) e^{-\frac{2m\pi i}{|\chi|}} \\
\left([k]_{\chi}^{-1}(\mathbb{Z}/|\chi|\,\mathbb{Z})^{\times} = (\mathbb{Z}/|\chi|\,\mathbb{Z})^{\times}\right);\ &= \sum_{m \in (\mathbb{Z}/|\chi|\mathbb{Z})^{\times}} \chi(m)\,\chi^{-1}(k)\, e^{-\frac{2m\pi i}{|\chi|}} \\
&= \chi^{-1}(k) \sum_{m \in (\mathbb{Z}/|\chi|\mathbb{Z})^{\times}} \chi(m)\, e^{-\frac{2m\pi i}{|\chi|}} \\
&= \chi^{-1}(k)\, G(\chi)
\end{aligned}$$

Thus:

$$\chi^{-1}(k) = \frac{\hat{\chi}(k)}{G(\chi)} = \frac{1}{G(\chi)} \sum_{n=0}^{|\chi|-1} \chi(n)\, e^{-\frac{2kn\pi i}{|\chi|}}$$

Taking conjugates gives us:

$$\chi(k) = \frac{1}{G(\chi^{-1})} \sum_{n=0}^{|\chi|-1} \chi^{-1}(n)\, e^{\frac{2kn\pi i}{|\chi|}} \tag{60}$$

So, using this formula in our integral formula for $L(\chi, s)$, we obtain:

$$\begin{aligned}
L(\chi, s) &= \frac{\beta^s}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{k=0}^{\beta-1} \frac{\chi(k) e^{kt}}{e^{\beta t} - 1} dt \\
&= \frac{\beta^s}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{k=0}^{\beta-1} \frac{\left(\frac{1}{G(\chi^{-1})} \sum_{n=0}^{\beta-1} \chi^{-1}(n) e^{\frac{2nk\pi i}{\beta}}\right) e^{kt}}{e^{\beta t} - 1} dt \\
&= \frac{\beta^s}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{G(\chi^{-1})} \sum_{n=0}^{\beta-1} \frac{\chi^{-1}(n)}{e^{\beta t} - 1} \sum_{k=0}^{\beta-1} \left(e^{t + \frac{2n\pi i}{\beta}}\right)^k dt \\
&= \frac{\beta^s}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{G(\chi^{-1})} \sum_{n=0}^{\beta-1} \frac{\chi^{-1}(n)}{e^{\beta t} - 1} \frac{\left(e^{t + \frac{2n\pi i}{\beta}}\right)^\beta - 1}{e^{t + \frac{2n\pi i}{\beta}} - 1} dt \\
&= \frac{\beta^s}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{G(\chi^{-1})} \sum_{n=0}^{\beta-1} \frac{\chi^{-1}(n)}{e^{\beta t} - 1} \frac{e^{\beta t} - 1}{e^{\frac{2n\pi i}{\beta}} e^t - 1} dt \\
&= \frac{\beta^s}{\Gamma(s)} \int_0^\infty t^{s-1} \frac{1}{G(\chi^{-1})} \sum_{n=0}^{\beta-1} \frac{\chi^{-1}(n)}{e^{\frac{2n\pi i}{\beta}} e^t - 1} dt
\end{aligned}$$

Thus, we are working with the function:

$$F_\chi(t) \overset{\text{def}}{=} -\frac{1}{G(\chi^{-1})} \sum_{n=0}^{\beta-1} \frac{\chi^{-1}(n)}{e^t e^{\frac{2\pi i n}{\beta}} - 1} \tag{61}$$

(the negative sign is needed, it turns out, to make things pretty). Using our $e^t = x + 1 \sim e^D = \Delta + 1$ analogy, writing:

$$\begin{aligned}
f_\chi\left(e^t - 1\right) &= F_\chi(t) \\
f_\chi(x) &= F_\chi(\ln(1 + x))
\end{aligned}$$

we have that the desired measure is:

$$dL_\chi \overset{\text{def}}{=} \frac{1}{-G(\chi^{-1})} \sum_{n=0}^{\beta-1} \frac{\chi^{-1}(n)}{(\Delta + 1) e^{\frac{2\pi i n}{\beta}} - 1} = \frac{1}{G(\chi^{-1})} \sum_{n=0}^{\beta-1} \sum_{k=0}^{\infty} \frac{\chi^{-1}(n) e^{\frac{2kn\pi i}{\beta}}}{\left(e^{\frac{2\pi i n}{\beta}} - 1\right)^{k+1}} \Delta^k$$

It can then be shown that, depending on the conductor of our dirichlet characters, we have the following formulae:

**$p$-adic $L$-functions—Measure-theoretic realization**: Let $\chi$ be any primitive Dirichlet character of conductor $p^n$ (where $p$ is prime and $n \in \mathbb{N}_1$), let $q$ be any positive integer co-prime to $p$, and let $\eta$ be any primitive Dirichlet character of conductor $q$. Then, we have the formulae:

$$\int_{\mathbb{Z}_p^\times} \chi(\mathfrak{y}) \mathfrak{y}^k \zeta_p(\mathfrak{y}) = L(\chi, -k), \ \forall k \in \mathbb{N}_1$$

and:

$$\int_{\mathbb{Z}_p^\times} \chi\left(\mathfrak{y}\right) \mathfrak{y}^k dL_\eta\left(\mathfrak{y}\right) = \left(1 - p^{k-1}\chi\left(p\right)\eta\left(p\right)\right) L\left(\chi\eta, -k\right), \ \forall k \in \mathbb{N}_1$$

where:

$$dL_\eta = \frac{1}{G\left(\eta^{-1}\right)} \sum_{n=0}^{q-1} \sum_{k=0}^{\infty} \frac{\eta^{-1}\left(n\right) e^{\frac{2kn\pi i}{q}}}{\left(e^{\frac{2\pi i n}{q}} - 1\right)^{k+1}} \Delta^k$$

Moreover, $\zeta_p$ and $dL_\eta$ are the unique measures on $\mathbb{Z}_p^\times$ for which these identities hold.

**From Measures to Analytic functions**

So far, we have seen how to construct *measures* on $\mathbb{Z}_p^\times$ that, when integrated against $\chi\left(\mathfrak{y}\right)\mathfrak{y}^k$, give us the value of $L$-functions at negative integers $-k$. Since the integers $k$ are dense in $\mathbb{Z}_p$, it then stands to reason that we can define our integral transforms as *analytic* functions by allowing $k$ to take on $p$-adic values $\mathfrak{s}$ by taking the $p$-adic limit our $p$-adic integrals as $k \to \mathfrak{s}$ in $\mathbb{Z}_p$. Indeed, formally, what we should get should be something like:

$$L\left(\chi, \mathfrak{s}\right) \overset{?}{=} \int_{\mathbb{Z}_p^\times} \chi\left(t\right) t^{-\mathfrak{s}} \zeta_p\left(t\right)$$

$$L\left(\chi\eta, \mathfrak{s}\right) \overset{?}{=} \frac{1}{1 - p^{\mathfrak{s}-1}\chi\left(p\right)\eta\left(p\right)} \int_{\mathbb{Z}_p^\times} \chi\left(t\right) t^{\mathfrak{s}} dL_\eta\left(t\right)$$

Alas, it is not that simple. In passing from the real variable $t$ in our mellin formulae to a $p$-adic variable $t$ in the integrals from the previous section, we unwittingly incurred the wrath of $e$. The $p$-adics have an insuperable enmity toward our beloved constant. Writing:

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!} = 1 + \frac{1}{2} + \frac{1}{2 \times 3} + \frac{1}{2 \times 3 \times 4} + \frac{1}{2 \times 3 \times 4 \times 5} + \cdots$$

we immediately find ourselves in a situation like that of $\zeta\left(s\right)$. The $n$th term of the power series defining $e$, $\frac{1}{n!}$, satisfies:

$$\left|\frac{1}{n!}\right|_p \leq \left|\frac{1}{\left(n+1\right)!}\right|_p$$

with:

$$\lim_{n \to \infty} \left|\frac{1}{n!}\right|_p = \infty$$

seeing as every time $n$ passes through a multiple of $p$, the largest power of $p$ dividing $n!$ increases by at least 1. Indeed: $e$ is *not* an element of $\mathbb{Z}_p$ for *any* $p$! (Worse yet, neither $\mathbb{Z}_p$, $\mathbb{Q}_p$, nor $\mathbb{C}_p$ possess any analogue of the identity $e^{2\pi i} = 1$!) The only consolation we have is that, unlike $\zeta\left(s\right) = \sum_{n=1}^{\infty} n^{-s}$, at least the power series for $e^{x}$ has $x^{n}$s in it, which, when sufficiently small, will provide enough of a decay to $\left|\frac{x^{n}}{n!}\right|_{p}$ to make it summable. As it turns out:

**Fact**: For $\mathfrak{a}\in\mathbb{C}_{p}$:

$$e^{\mathfrak{a}}=\sum_{n=0}^{\infty}\frac{\mathfrak{a}^{n}}{n!}$$

has a radius of convergence of $\left|\mathfrak{a}\right|_{p}<p^{-\frac{1}{p-1}}$. For $\mathfrak{a}\in\mathbb{Z}_{p}$, $e^{\mathfrak{a}}$ exists if and only if $\mathfrak{a}\in p\mathbb{Z}_{p}$. More generally:

$$t^{\mathfrak{s}}=e^{\mathfrak{s}\ln t}$$

is defined for all $\mathfrak{s}\in\mathbb{Z}_{p}$ and all $t\in p\mathbb{Z}_{p}$—but not necessarily for all $t\in\mathbb{Z}_{p}^{\times}$.

This shows us that our naïve attempt to extend our integral formulae to $p$-adic inputs was always doomed to failure. To make this work, we need to cheat by way of **Landau lifts/characters**. (Note: these objects are usually named after a student of Landau's. However, since their namesake is so repugnant an individual—he would have consigned me to die in a gas chamber in a Nazi concentration camp, had I lived in his era—and since he defamed and expelled his own doctoral advisor—Landau—I will consign his name to the cesspit of history. As far as I am concerned, the sooner it is forgotten, the better.)

**Definitions**:

I. Given $x\in\mathbb{Z}_{p}$, the **Landau Character** of $x$, denoted $\omega\left(x\right)$, is the unique $(p-1)$th root of unity (in $\mathbb{Z}_{p}$) integer defined by:

$$\omega\left(x\right)=\lim_{n\rightarrow\infty}x^{p^{n}} \tag{62}$$

where the limit is in $\mathbb{Z}_{p}$. We can think of $\omega$ as a map:

$$\omega:\mathbb{Z}_{p}\rightarrow\{0\}\cup\{p-1\text{ roots of unity in }\mathbb{Z}_{p}\}$$

where $\omega\left(x\right)\neq0$ if and only if $x\in\mathbb{Z}_{p}^{\times}$.

II. Defining the map $\left\langle \cdot\right\rangle :\mathbb{Z}_{p}^{\times}\rightarrow1+p\mathbb{Z}_{p}$ by:

$$\left\langle x\right\rangle \overset{\text{def}}{=}\frac{x}{\omega\left(x\right)}\in1+p\mathbb{Z}_{p} \tag{63}$$

we can then write every $x\in\mathbb{Z}_{p}^{\times}$ as a product:

$$x=\omega\left(x\right)\times\left\langle x\right\rangle \tag{64}$$

called the **Landau decomposition**. This decomposition gives us an isomorphism of multiplicative groups:

$$\left(\mathbb{Z}_{p}^{\times},\times\right)\cong\mathbb{Z}/(p-1)\mathbb{Z}\times\left(1+p\mathbb{Z}_{p}\right)$$

With this construction, we can then define a convergent form of $p$-adic exponentiation of $t \in \mathbb{Z}_p^\times$ by $-\mathfrak{s} \in \mathbb{Z}_p$ by the map $t \mapsto \omega(t) \times \langle t \rangle^{-\mathfrak{s}}$. s a consequence of the Landau character, it turns out that the $p$-adic analogue of $\zeta(s)$, is a multi-valued function for all odd $p$. For every $p$, it has $p-1$ branches, indexed by $j \in \{1, \ldots, p-1\}$, corresponding to the $p-1$ maps:

$$t \mapsto (\omega(t))^{-j} \langle t \rangle^{-\mathfrak{s}}$$

Consequently, we can write:

**Definition**: The $j$th branch of the ***p*-adic zeta function** $\zeta_{p,j} : \mathbb{Z}_p \to \mathbb{C}_p$ is given by:

$$\zeta_{p,j}(\mathfrak{s}) \overset{\text{def}}{=} \int_{\mathbb{Z}_p^\times} (\omega(t))^{-j} \langle t \rangle^{-\mathfrak{s}} \zeta_p(t), \ \forall \mathfrak{s} \in \mathbb{Z}_p, \forall j \in \{1, \ldots, p-1\} \tag{65}$$

*Remark*: the *function* $\zeta_{p,j}(\mathfrak{s})$ does not give us as many values of $\zeta(s)$ as can be obtained by the *measure* $\zeta_p$. Because of our use of the Landau character, we note that for any integer $k$ and any $t \in \mathbb{Z}_p$, the $p$-adic number $t^k$ has a Landau decomposition:

$$t^k = (\omega(t))^j \langle t \rangle^k$$

if and only if $k \overset{p-1}{\equiv} j$.

That being said, we do have:

**Theorem**: for all $j \in \{1, \ldots, p-1\}$ and all $k \in \mathbb{N}_0$ with $k \overset{p-1}{\equiv} j$:

$$\zeta_{p,j}(-k) = \left(1 - p^k\right) \zeta(-k) \tag{66}$$

*Remark*: Notice that, replacing $k$ with $n-1$ turns the right-hand side into $\left(1 - p^{n-1}\right) \zeta(1-n)$, which is exactly the expression from Kummer's congruences for the case where $p$ is an odd prime:

$$\left(1 - p^{m-1}\right) \zeta(1-m) \overset{p^{a+1}}{\equiv} \left(1 - p^{n-1}\right) \zeta(1-n)$$

The definition of $p$-adic $L$-functions then follows exactly as we would expect:

**Definition**: Let $\theta$ be a dirichlet character with factorization $\theta = \chi\eta$ where $|\chi| \in p^{\mathbb{N}_0}$ and $\gcd(|\eta|, p) = 1$. Then, the ***p*-adic *L*-function** of $\theta$ is:

$$L_p(\theta, \mathfrak{s}) \overset{\text{def}}{=} \int_{\mathbb{Z}_p^\times} \chi(t) (\omega(t))^{-1} \langle t \rangle^{-\mathfrak{s}} dL_\eta(t) \tag{67}$$

*Remark*: We have the identity:

$$\zeta_{p,j}(\mathfrak{s}) = L_p\left(\omega^{j+1}, \mathfrak{s}\right) \tag{68}$$

Just like with the $p$-adic zeta function, we also have an analogue of Kummer's congruences for the $p$-adic $L$-functions:

**Theorem**:

$$L_p(\theta, 1-k) = \left(1 - \theta(p) (\omega(p))^{-k} p^{k-1}\right) L\left(\theta\omega^{-k}, 1-k\right), \ \forall k \geq 1$$

## **Appendix**:

List of the Author's Idiosyncratic Notations

- $\mathbb{R}^+ \overset{\text{def}}{=} \{x \in \mathbb{R} : x > 0\}$
- For any $x \in \mathbb{R}$, $\mathbb{N}_x \overset{\text{def}}{=} \{n \in \mathbb{Z} : n \geq x\}$.
- For any $n \in \mathbb{N}_1$, $\xi_n \overset{\text{def}}{=} e^{\frac{2\pi i}{n}}$. When used in a $p$-adic context, $\xi_n$ denotes the $n$th root of unity in the $p$-adic space corresponding to $\xi_n$ under an embedding.
- $\mathbb{P}$ is the set of prime numbers: $2, 3, 5, 7, 11, \ldots$
- Operators whose output is a function/map/input-acceptor are written:

$$\text{Operator}\,\{\text{input}\}\,(\text{input for the output})$$

For example, $\mathscr{M}\{f\}(s)$ is the Mellin transform of $f$ evaluated at $s$ while $\mathscr{M}\{f\}$ is the Mellin transform of $f$ (a function waiting to be evaluated).

- $a \overset{r}{\equiv} b$ means "$a$ is congruent to $b$ modulo $r$"
- Specific groups are occasionally denoted by $(\text{set}, \text{operation})$. So, for instance $(\mathbb{R}^+, \times)$ is the group obtained by equipping $\mathbb{R}^+$ with the multiplication operation, while $(\mathbb{Q}_p, +)$ is the group of $p$-adic rational numbers under the addition operation.